APPM 2360: Midterm Exam 3

November 16, 2016

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized **1 side only**) crib sheet is allowed.

Problem 1: (30 points; 6 points each) True/False questions. Answer True if the statement is always true. Answer False otherwise. Box your answer. No partial credit will be given.

(a) If $y(t) \neq 0$ is a solution to the homogeneous mass-spring problem $my'' + by' + ky = 0$ with damping (i.e. $b > 0$), then there is at most one value of t where $y(t) = 0$.

(b) Let x be a solution to $x'' + 9x = \cos(3t)$. Then the amplitude of x is bounded for all time.

(c) $y_1(t) = e^{-2t} + e^{-3t}$ and $y_2(t) = e^{-2t} - e^{-3t}$ are linearly independent solutions to $y'' + 5y' + 6y = 0$.

(d) *All* solutions to $y'' + 4y = 0$ can be written in the form $y(t) = A\cos(2t - \delta)$ for some choice of constants A and δ.

(e) Consider the DE:

$$y''' + y'' + 2y' + 3y = 0. \tag{1}$$

If $y_1(t)$ and $y_2(t)$ are two linearly independent solutions to Equation 1, then *every* possible solution to Equation 1 may be written as $y(t) = c_1 y_1(t) + c_2 y_2(t)$ for an appropriate choice of the constants c_1 and c_2.

Problem 2: (30 points) Consider the DE

$$y'' + 4y' + 4y = f(t) \tag{2}$$

(a) Find the general solution to the homogeneous system ($f(t) = 0$).

(b) Using the method of undetermined coefficients, find the **FORM** of a particular solution of Equation 2 when

(i) $f(t) = f_1(t) = e^t + 3\sin(t)$,

(ii) $f(t) = f_2(t) = 7e^{-2t}$.

Please do not solve for the coefficients.

(c) Using the method of undetermined coefficients, find a particular solution of Equation 2 if

$$f(t) = f_3(t) = 32te^{2t}.$$

You should solve for the coefficients.

Problem 3: (30 points) Consider the second order DE for $t > 0$:

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = f(t). \tag{3}$$

Note that $y_1(t) = t$ and $y_2(t) = t^2$ are both solutions to Equation 3 when $f(t) = 0$.

(a) Compute the Wronskian of y_1 and y_2

(b) Use the method of variation of parameters to find a particular solution to Equation 3 when $f(t) = 15t^{3/2}$.

(c) Write down the general solution of Equation 3, using the $f(t)$ from part (b).

*****TEST CONTINUES ON OTHER SIDE OF PAGE*****

Problem 4: (30 points) Consider the general mass-spring system described by

$$mx'' + bx' + kx = f(t),$$

with mass $m = 2$ kg and spring constant $k = 32$ nt/m. $f(t)$ is some forcing function, and b is the damping constant.

(a) In the absence of forcing and damping, what is the resonant frequency of the system?

(b) Suppose that damping is now present ($b > 0$) and that no forcing is applied. We displace the mass a distance of 1m from its equilibrium position and release it. What value(s) of the damping constant b will ensure that the mass crosses its equilibrium position *no more than once*?

(c) Describe the long-term behavior of the undamped system when it is subject to the forcing function $f(t) = \cos 4t$. **Note:** Simply writing down a formula for the general solution does not constitute an answer to this question.

(d) Find the unique solution $x(t)$ for the undamped, unforced system subject to the initial conditions $x(0) = 10$, $x'(0) = 0$.

Problem 5: (30 points)

(a) (25 points) Consider the initial value problem

$$y'' + y' - 6y = 1, \quad y(0) = -1, \quad y'(0) = 0.$$

Solve for y using the Laplace transform method; *do not use other methods!*

(b) (5 points) What is the inverse Laplace transform of $X(s) = \frac{1}{(s+4)^2+36}$?

Table of Laplace Transforms

$$\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)dt$$

By linearity, $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$.

1. $\mathcal{L}\{1\} = \dfrac{1}{s}, \; s > 0$

2. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, n a positive integer, $s > 0$

3. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \; s > a$

4. $\mathcal{L}\{t^n e^{at}\} = \dfrac{n!}{(s-a)^{n+1}}$, n a positive integer, $s > a$

5. $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}, \; s > 0$

6. $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}, \; s > 0$

7. $\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}, \; s > a$

8. $\mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}, \; s > a$

9. $\mathcal{L}\{\sinh bt\} = \dfrac{b}{s^2-b^2}, \; s > |b|$

10. $\mathcal{L}\{\cosh bt\} = \dfrac{s}{s^2-b^2}, \; s > |b|$

11. $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}F(s), \; s > 0$

12. $\mathcal{L}\{e^{at}f(t)\} = F(s-a), \; s > a$

13.* $\mathcal{L}\{f'(t)\} = sF(s) - f(0), \; s > \alpha$

14.* $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0), \; s > \alpha$

15.* $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0), \; s > \alpha$ (nth derivative)

*α is the exponential order of $f(t)$ and its derivatives. [See Sec. 8.3 equation (4).]