APPM 2360: Midterm exam 2, SOLUTIONS

October 19, 2016

Problem 1: (24 points) **True/False** (answer True if it is always true; otherwise answer False) No justification is needed.

(a) The system of equations $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has at least one solution $\vec{x}$ for any values of a, b, c, d, e, f.

(b) The polynomials $\{x^2 + 1, x + 2, x^2 - x - 1\}$ form a basis for $\mathbb{P}_2$.

(c) If $\vec{v}_i$, $i = 1, 2, \ldots 5$ are vectors, and $\mathbb{W} = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$, then the dimension of $\mathbb{W}$ is always 5.

(d) The set of all 4×4 matrices that have 0 entries on the diagonal is a vector space.

Solution:

(a) False. The equation may be inconsistent.

(b) False. These polynomials are not linearly independent. Note that

$$x^2 - x - 1 = \left(x^2 + 1\right) - (x + 2)\,. \tag{1}$$

(c) False. The dimension of $\mathbb{W}$ can be 0,1,2,3,4 and 5.

(d) True, it satisfies the closure properties under addition and scalar multiplication.

Problem 2: (24 points) **Short Answer** questions. A short justification is sufficient but *do not* simply answer True/False or Yes/No.

(a) Are the vectors $[0, 1, 1], [0, 2, 0], [2, 0, 1] \in \mathbb{M}_{13}$ (the space of 1×3 matrices) linearly independent or linearly dependent?

(b) Consider the set of all solutions to the differential equation

$$y'' + y = \sin(2t).$$

Is this set a vector space? (Remember to briefly justify your answer).

(c) Consider the set

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ k \end{bmatrix} \right\}.$$

(i) For what value(s) of k does W constitute a basis for $\mathbb{R}^3$?

(ii) If $\mathbb{V} = \text{Span}(W)$, for what value(s) of k is $\dim(\mathbb{V}) = 2$?

Solution:

(a) Yes, these are linearly independent. You can put these into a system of equations and check that the coefficient matrix is invertible (e.g.., check the determinant is nonzero, or that you have 3 pivot columns after row-reducing).

(b) No, since this is an *inhomogenous* linear ODE. For example, $y(t) = 0$ is not a solution, so it cannot be a vector space.

(c) (i) $k \neq 0$ makes this set of three vectors linearly independent. The question wanted all possible values of k.

(ii) $k = 0$ makes this set linearly dependent (the third vector is the sum of the first two). In this case, only the first two vectors are needed to form a basis for $\mathbb{V}$, giving it dimension 2.

Problem 3: (26 points) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}.$$

(a) Write down the characteristic polynomial that determines eigenvalues of A.

(b) Find the eigenvalues of A.

(c) Find the eigenvector(s) of A for its **largest** eigenvalue. (only the LARGEST λ!)
(d) What is the dimension of the eigenspace $\mathbb{E}_\lambda$ for the eigenvalue in (c)?

Solution:

(a) Expanding along the second row, then

$$p(\lambda) = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix}$$
$$= (1-\lambda)(\lambda^2-\lambda) = -\lambda(1-\lambda)^2$$

(b) $p(\lambda) = -(\lambda-1)^2(\lambda-0)$. So the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1$ and $\lambda_3 = 0$ (you should list the 1 eigenvalue twice, or make specific mention that it has algebraic multiplicity 2).

(c) To find the eigenvector(s) for $\lambda = 1$, we solve

$$(A-I)\vec{v} = 0$$

which gives the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

As the second row is zero and the last row is independent of the first there is going to be one free variable. Flipping row 2 and 3 and subtracting twice the first row from the (new) 2nd row gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which implies that the eigenvector solves $v_3 = s$ (arbitrary), $v_2 = 0$, and $v_1 = v_3 = s$, so we have

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Note: any other non-zero value of s is fine here.

(d) The dimension of $\mathbb{E}_1$ is 1 since there is one eigenvector for $\lambda = 1$. (Note that this matrix has a deficiency of eigenvectors; it is not diagonalizable).

Problem 4: (24 points)

(a) Consider the following two matrices:

$$A = \begin{bmatrix} 1 & 4 & 7 & 9 \\ 0 & 2 & 11 & 12 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

For each matrix, indicate 1) its rank and 2) whether its inverse exists. Do not calculate the inverse; only indicate whether or not it exists. In each case, explain your answer and/or show your work.

Solution: Note that since A is upper triangular, and the diagonal elements are nonzero, it has four nonzero pivots and so has rank 4 (indeed $|A| = 4$ as well), thus it has an inverse. B has the RREF

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so it has rank 2. Since this is not full rank, it has no inverse.

(b) Consider the following system of equations $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ -1 \\ k \end{bmatrix}$.

(i) For which value(s) of k is the system consistent?
(ii) For a value of k for which the system is consistent, determine at least **two** solutions.

Solution:

(a) The matrix is not invertible (not full rank), so there is not always a solution. The 3rd equation is twice the 2nd equation, so we need $k = -2$. To see it more systematically, put into RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 6 & k \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & k+2 \end{array}\right]$$

so this last equation is only possible if $k + 2 = 0$, i.e., $k = -2$.

(b) For $k = -2$, we get that x_3 is a free variable, so call this $x_3 = t$ for any $t \in \mathbb{R}$. Then the first equation requires $x_1 + 2t = -2$ and the next equation requires $x_2 + 3t = -1$, so our general solution is $\vec{x} = \begin{bmatrix} 2 - 2t \\ -1 - 3t \\ t \end{bmatrix}$. Plug in any values of t to find solutions, e.g., $t = 1$ gives $\begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}$; and $t = 0$ gives $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Problem 5: (24 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 4 & 3 & 5 & 2 \end{bmatrix}$

(a) Find the RREF of A.

(b) Find a basis for

$$\mathbb{U} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

(Notice these are the column vectors of the matrix A.)

(c) Are the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 5 \end{bmatrix}$ linearly independent? Explain. (Hint: they are column vectors of A).

Solution:

(a) $$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 4 & 3 & 5 & 2 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3, R_4^* \leftarrow R_4 - R_3^*]{R_1^* \leftarrow R_1 - R_2 - R_3} \begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is the RREF of A.

(b) $U = Col(A)$, from (a), we see that columns $1, 3, 5$ are the pivot columns, and the columns 2 and 4 are free. Consequently , the three vectors of columns $1, 3, 5$ of A: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis of U.

(c) Dim(Col(A))=3 and here are 4 column vectors of A, they are **linearly dependent**.

Problem 6: (28 points) Consider the linear system:

$$\begin{array}{rcrcrcr} 2x & + & y & & & = & 0 \\ 5x & + & 3y & & & = & 1 \\ & - & 2y & + & z & = & -4. \end{array}$$

(a) Write this system in matrix form as $A\vec{x} = \vec{b}$.

(b) Compute A^{-1}.
(Hint: you can check your answer by testing if $A^{-1}A = \mathbf{I}$.)

(c) Use your answer from part (b) to compute $\vec{x}$. Do not row-reduce to solve for $\vec{x}$.

Solution:

(a)

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} \tag{2}$$

(b) To find A^{-1}, we augment A with the indentity matrix and reduce A to its RREF:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 5 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2^* = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1^* = R_1 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 1 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2^* = R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -5 & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3^* = R_3 + 2R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & -5 & 2 & 0 \\ 0 & 0 & 1 & -10 & 4 & 1 \end{array}\right].$$

We see that $A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -5 & 2 & 0 \\ -10 & 4 & 1 \end{bmatrix}$.

(c) We solve for $\vec{x}$ by evaluating $\vec{x} = A^{-1}\vec{b}$, and find that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ -5 & 2 & 0 \\ -10 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \tag{3}$$