APPM 2360: Final Exam

December 13, 2016

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your instructor's name, (3) your recitation section number and (4) a grading table. Text books, class notes, cell phones and calculators are NOT permitted. A one page (letter sized, **2 sided**) crib sheet is allowed. Start each problem on a new page

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Problem 1: (30 points) True/False questions. Answer "True" if the statement is *always* true, otherwise answer "False." Box your answer. No partial credit will be given.

(a) The solution space of $3y''' + y'' - y = 0$ is a vector space of dimension 3.

(b) The Laplace transform is a linear operator.

(c) For any (possibly non-linear) second-order differential equation, we can write *all* solutions as a linear combination of two linearly independent solutions.

(d) All nonzero solutions to $y'' + 4y' + 8y = 0$ cross the $y = 0$ axis at most once.

(e) The polynomials $\{x^2 + 1, x + 2, 2x^2 - x - 1\}$ form a basis for $\mathbb{P}^2$

(f) Let x be a solution to $x'' + 9x = \cos(2t)$. Then x is bounded for all time.

Problem 2: (30 points) Short answer. Remember to briefly **justify your answer**.

(a) If A is an $m \times n$ matrix (m rows, n columns), when are there *infinitely* many solutions to the linear system $A\vec{x} = 0$?

(b) If A and B are square matrices, then does $|AB| = |BA|$? Explain.

(c) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis for a vector space $\mathbb{V}$, what is the dimension of $\mathbb{V}$?

(d) Recall that $\mathbb{M}_{22}$ is the vector space comprised of all of 2×2 matrices. Suppose that $\mathbb{W}$ is defined as the set of all 2×2 matrices whose first element $\mathrm{a}_{11} > 0$. Demonstrate that $\mathbb{W}$ **does not** form a vector subspace of $\mathbb{M}_{22}$.

(e) Suppose $y' = 2t^3 + f(t)y^2$. For what functions $f(t)$, if any, is this DE separable?

(f) Consider the set of all solutions to the logistic equation

$$\frac{dy}{dt} = 2\,(1 - y)\,y.$$

Is this set a vector space?

Problem 3: (40 points) Consider the initial value problem:

$$y' - 2ty = 6te^{t^2}, \quad y(0) = 2016. \tag{1}$$

(a) Find a nontrivial solution to the homogeneous equation.

(b) Find a particular solution to the differential equation in (1).

(c) Determine the general solution to the differential equation.

(d) Determine the solution to the initial value problem.

Problem 4: (35 points)

(a) Consider the linear system

$$\begin{array}{rcrcrcl} 10x & + & 4y & + & z & = & a \\ 5x & + & 3y & & & = & b \\ & - & 2y & + & z & = & c, \end{array}$$

where a, b, and c are any real numbers. Does this system of equations have a unique solution? **Justify your answer**.

*****TEST CONTINUES ON OTHER SIDE OF PAGE*****

(b) Solve the following linear system using row-reduction:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 9 \\ x & + & 2y & + & 2z & = & 17 \\ x & + & y & + & 2z & = & 14 \end{array}$$

(c) Consider the set

$$U = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Is U a basis for $\mathbb{R}^3$ (the set of all 3-vectors)? Justify your answer.

Problem 5: (40 points)

(a) Find a *particular solution* for the following equation using the method of undetermined coefficients (**solve** for any constants):

$$y'' + y = \sin(2t).$$

(b) Find the *form* of a particular solution for the following equation (do **not** solve for any constants):

$$y'' + 3y' + 2y = e^{-2t}.$$

(c) Find functions $v_1(t)$ and $v_2(t)$ such that $y(t) = v_1(t)e^{-2t} + v_2(t)e^{3t}$ is a solution to

$$y'' - y' - 6y = 5.$$

Problem 6: (35 points)

(a) Find the inverse Laplace transform of $X(s) = \frac{e^{-3s}}{s^2+1}$.

(b) Consider the initial value problem

$$y'' + y = \delta(t-3), \quad y(0) = 5, \quad y'(0) = 0.$$

Solve for y using the Laplace transform; *do not use other methods!*

Problem 7: (40 points)

(a) Consider the following system of DEs

$$x' = 2 - x + y$$
$$y' = y^2 - x$$

(i) What are the horizontal and vertical nullclines?
(ii) What are the equilibria?
(iii) Classify the equilibria: are they stable or unstable? You may want to create a phase plane graph to help answer this question (e.g., graph the nullclines, with a few more representative direction field vectors), but your graph will not be graded.

(b) Consider the system of two first order DEs given by

$$\frac{d}{dt}\vec{x} = A\vec{x}, \quad A = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}. \tag{2}$$

(i) Find the eigenvalues and eigenvectors of A.
(ii) Find the general solution of the the two first order DEs listed above in (2).
(iii) Is the equilibrium for this system stable or unstable?

Table of Laplace Transforms. $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$ where f is of exponential order α

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s),\ s > 0$	$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2},\ s > 0$	$\mathcal{L}\{\delta(t)\} = 1,\ s > 0$
$\mathcal{L}\{e^{at}f(t)\} = F(s-a),\ s > a$	$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2},\ s > 0$	$\mathcal{L}\{f'(t)\} = sF(s) - f(0),\ s > \alpha$
$\mathcal{L}\{\text{step}(t)\} = \frac{1}{s},\ s > 0$	$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ s > 0$	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0),\ s > \alpha$
$\mathcal{L}\{f(t-a)\text{step}(t-a)\} = e^{-as}F(s),\ s > a$	$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s > a$	