Problem 1: (36 points, 6 points each) **True/False** (answer True if it is always true otherwise answer False) or **Short Answer** for the following problems. No justification is needed.

(a) The differential equation $y' + y\sin^2 t = t^2 y + 1 - t^2 - y\cos^2 t$ is separable. (True/False)

(b) Picard's Theorem tells us that the IVP $y' = t\sqrt{y}$, $y(1) = 0$ has a unique solution. (True/False)

(c) Consider the logistic equation $p' = 2(1 - \frac{p}{100})p$. If $p_1(t)$ and $p_2(t)$ are both solutions to the equation, then $p(t) = p_1(t) + p_2(t)$ is always a solution. (True/False)

(d) Given the fact that $y(t) = e^{2t}$ is a solution to the differential equation

$$y'(t) + p(t)y(t) = 2e^{2t} + e^{3t}.$$

Find the function $p(t)$. (Short Answer)

(e) Consider the coupled system of equations

$$\frac{dx}{dt} = 9x - 3xy$$
$$\frac{dy}{dt} = -2y + xy.$$

Find the vertical nullcline(s) of this system. (Short Answer)

(f) For the following differential equation

$$\frac{dy}{dt} = y(3 - y),$$

find all equilibrium solutions and classify them as stable, unstable or semistable. (Short Answer)

Solution:

(a) True. We may rewrite this equation as $y' = (t^2 - 1)(y - 1)$.

(b) False. This DE fails the uniqueness test at $y(1) = 0$ where $\frac{\partial f}{\partial y} = \frac{t}{2\sqrt{y}}$ is discontinuous.

(c) False, the equation is not linear due to the p^2 term.

(d) $p(t) = e^t$.

(e) Solve for $x' = 0$ to get $9x = 3xy$, i.e., $\boxed{x = 0}$ and $\boxed{y = 3}$.

(f) $y = 0$ is unstable, and $y = 3$ is stable.

Problem 2: (30 points) Consider the differential equation

$$\frac{dy}{dt} = -2y\sin t - 2\sin t \tag{1}$$

(a) Find the general solution to Eq. (1) using separation of variables.

(b) Demonstrate that your solution from (a) indeed satisfies the differential equation (1).

(c) Find the unique solution to Eq. (1) that passes through $(t = \pi/2, y = 5)$

(d) What is the nature of the solution that passes through $(t = 0, y = -1)$?

Solution:

(a) • We separate variables and find that

$$\frac{dy}{y+1} = -2\sin t dt. \tag{2}$$

• Integrating both sides, we find that

$$\ln|y+1| = 2\cos t + C, \tag{3}$$

where C an undetermined constant. Solving for y, we find that

$$y = \pm e^C e^{2\cos t} = Be^{2\cos t} - 1, \tag{4}$$

where the undetermined constant B may be any *nonzero* real number. By continuity, or by checking for the equilibrium solution, we relax the constraint so that we allow $B = 0$ as well, and arrive at the general solution

$$y = Be^{2\cos t} - 1, \quad B \in \mathbb{R}. \tag{5}$$

(b) We check our answer by differentiating our solution, finding

$$y' = (-2\sin t)\, Be^{2\cos t}. \tag{6}$$

We rewrite the exponential in terms of y to see that

$$y' = -2\sin t(Be^{2\cos t} - 1 + 1) = -2\sin t(y+1), \tag{7}$$

Or $-2\sin t(y+1) = -2\sin t(Be^{2\cos t} - 1 + 1) = -2\sin tBe^{\cos t} = y'$. which is just our original DE.

(c) The solution that passes through $(\pi/2, 5)$ corresponds to $B = 6$, yielding

$$y = 6e^{2\cos(t)} - 1. \tag{8}$$

(d) The solution that passes through (0,-1) corresponds to the equilibrium (or steady, or stationary, or constant) solution $y = -1$, which is constant for all t. *(6 points)*

Problem 3: (30 points) Consider the initial value problem:

$$ty' + (t^2+1)y = te^{-t^2}, \quad y(2) = 0.$$

(a) Find the solution to the homogeneous equation.
(b) Using the variation of parameters method, find a particular solution.
(c) Determine the general solution to the differential equation.
(d) Determine the solution to the initial value problem.

Solution: Note: Must put equation in standard form to use V. of P.

$$y' + (t + \frac{1}{t})y = e^{-t^2}.$$

(a) $y_h = \frac{c}{t}e^{-t^2/2}$
(b) Must solve $v'(t)\frac{1}{t}e^{-t^2/2} = e^{-t^2}$. Then $v(t) = -e^{-t^2/2}$ giving $y_p = -\frac{1}{t}e^{-t^2}$
(c) $y_g = -\frac{1}{t}e^{-t^2} + \frac{c}{t}e^{-t^2/2}$.
(d) For $y(2) = 0$, $c = e^{-2}$, so

$$y = \frac{1}{t}\left(\frac{1}{e^2} - e^{-t^2/2}\right)e^{-t^2/2}$$

Problem 4: (30 points) [Note: if your answer involves logarithms, you may leave these unevaluated]

(a) A scientist begins an experiment several years ago starting with 32/9 grams of a radioactive substance. Last year, only 2 grams of the substance remained, and this year (exactly 1 year later), only 1.5 gram of the substance remain. How many years ago did the scientist begin the experiment?

(b) What is the half-life of the radioactive substance?

Solution:

(a) The main trick is to set $t = 0$ to be either last year or this year, and *not* the unknown time when the experiment started.

If we set $t = 0$ to be last year, then $y(t) = 2e^{-kt}$ for some k. Using units of year for the time, then using data from this year lets us solve for k: $1.5 = 2e^{-k}$ so $-k = \ln(3/4)$ or $\boxed{k = \ln(4/3)}$. Then solve for T, which is the number of years (relative to last year) when there was 32/9 grams. A negative T will indicate years in the past (before last year)/

We set $32/9 = 2e^{-kT} = 2(e^{-k})^T = 2(3/4)^T$, so we want to solve $16/9 = (3/4)^T$ i.e., $16/9 = (4/3)^{-T}$. From here, a valid answer is $\boxed{T = \log_{3/4}(16/9) = \ln(16/9)/\ln(3/4)}$, but you can also observe by inspection that $\boxed{T = -2}$. Hence the answer is "two years before last year", i.e., $\boxed{\text{"three years ago."}}$

ALTERNATIVE SETUP: If we set $t = 0$ to be this year, then $y(t) = 1.5e^{-kt}$, and to solve for k, we use last year's data: $2 = 1.5e^{+k}$ so we again find $k = \ln(4.3)$. Then solve $32/9 = 3/2e^{-kT}$ which is the same as $64/27 = (3/4)^T$ and by inspection, since $4^3 = 64$ and $3^3 = 27$, we have $T = -3$, so our answer is "three years ago."

(b) Solve $1/2 = e^{-kT}$ where T is the half-life. Take the log of both sides and use the value of k we just found to get $\ln(1/2) = -kT$ or $\ln(1/2) = -\ln(4/3)T$ so

$$\boxed{T_{1/2} = -\ln(1/2)/\ln(4/3) = \ln(2)/\ln(4/3) = \ln(1/2)/\ln(3/4) = -\ln(2)/\ln(3/4) \approx 2.409}$$

in units of years (any of those are acceptable).

Problem 5: (24 points) Suppose that a tank contains 100 gallons of water with an initial salt concentration of 5 oz/gal. A solution with a concentration of 10 oz/gal of salt is added at a rate of 5 gal/min and the well-stirred mixture drains from the tank at the same rate.

(a) Set up an initial-value problem describing the amount of salt in the tank after t minutes.
(b) Find the solution to this IVP.
(c) What is the long-term behavior of this solution?

Solution:

(a) $Q(0) = 5\frac{oz}{gal}100gal = 500oz$, rate in $= 10 \times 5 = 50\frac{oz}{min}$, rate out$= 5 \times \frac{Q}{100} = \frac{Q}{20}$.
The IVP is: $\mathbf{\frac{dQ}{dt} = 50 - \frac{Q}{20} = \frac{1}{20}(1000 - Q), Q(0) = 500}$.

(b) Let $x = Q - 1000$, then $\frac{dx}{dt} = -\frac{1}{20}x, x(0) = Q(0) - 1000 = 500 - 1000 = -500$. $x(t) = x(0)e^{-\frac{1}{20}t} = -500e^{-\frac{1}{20}t}$.
$\mathbf{Q(t) = 1000 + x(t) = 1000 - 500e^{-\frac{1}{20}t}}$.

(c) As $t \longrightarrow +\infty$, $Q(t) \longrightarrow$ **1000oz**.