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Well-posedness of a 3D parabolic–hyperbolic Keller–Segel system in the Sobolev space framework

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Abstract

We investigate global strong solution to a 3-dimensional parabolic–hyperbolic system arising from the Keller–Segel model. We establish the global well-posedness and asymptotic behavior in the energy functional setting. Precisely speaking, if the initial difference between cell density and its mean is small in L^2, and the ratio of the initial gradient of the chemical concentration and the initial chemical concentration is also small in H^1, then they remain to be small in $L^2 \times H^1$ for all time. Moreover, if the mean value of the initial cell density is smaller than some constant, then the cell density approaches its initial mean and the chemical concentration decays exponentially to zero as t goes to infinity. The proof relies on an application of Fourier analysis to a linearized parabolic–hyperbolic system and the smoothing effect of the cell density and the damping effect of the chemical concentration.

MSC: 35Q92; 35B40; 35G55; 92C17

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1. Introduction

In this paper, we study the following three-dimensional (3D) chemotaxis model

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$$\begin{cases} \partial_t u = \Delta u + \nabla \cdot (u \nabla \ln v), \\ \partial_t v = uv - \mu v \end{cases} \tag{1.1}$$

where $(t, x) \in (0, \infty) \times \mathbb{R}^3$, $u(t, x)$ and $v(t, x)$ denote the cell density and the chemical concentration respectively, and μ is a constant. The initial data is

$$(u, v)(0, x) = (u_0, v_0)(x) \tag{1.2}$$

for $x \in \mathbb{R}^3$. System (1.1) was proposed by Othmer and Stevens [26] to describe the chemotactic movement of particles where the chemicals are non-diffusible and can modify the local environment for succeeding passages. For example, myxobacteria produce slime over which their cohorts can move more readily and ants can follow trails left by predecessors [8]. One direct application of (1.1) is to model haptotaxis where cells move towards an increasing concentration of immobilized signals such as surface or matrix-bound adhesive molecules.

For the sake of simplicity, we set $w = \mu t + \ln v$. Therefore we get from (1.1) that

$$\begin{cases} \partial_t u = \Delta u + \nabla \cdot (u \nabla w), \\ \partial_t w = u \end{cases} \tag{1.3}$$

where $(t, x) \in (0, \infty) \times \mathbb{R}^3$. (1.3) is supplemented with initial data $(u_0, w_0)(x) = (u_0, \ln v_0)$ for all $x \in \mathbb{R}^3$.

It is worth mentioning that (1.3) was studied in [30] for the 1-dimensional case and was extended to multidimensional cases in [20,21]. It was studied in [26] and a comprehensive qualitative and numerical analysis were provided. We refer readers to Refs. [3,5,6,8,10–13,18–20, 22–25,27,28,30–33] for more discussions in this direction. Recently, in [20], the local and global existence of the classical solution to (1.3) in 3-dimension were studied when[1]

$$(u_0 - \bar{u}, \nabla w_0) \in H^{\frac{5}{2}+} \times H^{\frac{5}{2}+}.$$

Here H^s is the Bessel potential space

$$H^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^3);\ \left\| (1 - \Delta)^{s/2} f \right\|_{L^2} < \infty \right\} \tag{1.4}$$

Throughout this paper, we shall omit the space domain $\mathbb{R}^3$ for the sake of simplicity such that $X(\mathbb{R}^3)$ is denoted by X for any given Banach space X. If the space dimension is not 3, then we will indicate the space domain.

Later on, Hao [9] studied existence and uniqueness of global mild solutions for initial data close to some constant state in critical Besov space with minimal regularity. The proof was in the Chemin–Lerner space framework (see e.g. [3,7]).

The Cauchy problem of (1.3) is invariant under the following scaling transformations:

$$(u, w, u_0, w_0) \longrightarrow (u_\lambda, w_\lambda, u_{0\lambda}, w_{0\lambda}),$$

where $u_\lambda(t, x) = \lambda^2 u(\lambda^2 t, \lambda x)$, $w_\lambda(t, x) = w(\lambda^2 t, \lambda x)$, $u_{0\lambda}(x) = \lambda^2 u_0(\lambda x)$ and $w_{0\lambda}(x) = w_0(\lambda x)$. The idea of using an invariant functional setting was originated from many works (see

[1] $\bar{u}$ denotes the mean value of u_0 and $s = \frac{5}{2}+$ stands for $s > \frac{5}{2}$. Similar conventions are applied.

e.g. [2]). In this paper, we shall also employ the invariant argument. As a consequence, we observe that the critical Sobolev space for (u_0, w_0) is $\dot{H}^{-\frac{1}{2}} \times \dot{H}^{\frac{3}{2}}$ and the subcritical Sobolev space is $\dot{H}^{-\frac{1}{2}+} \times \dot{H}^{\frac{3}{2}+}$ which is also called the Riesz potential space

$$\dot{H}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^3);\ \left\| \Lambda^s f \right\|_{L^2} < \infty \right\}. \tag{1.5}$$

In the critical case, it seems difficult to prove the global existence of the solution to (1.3) with $(u_0, w_0) \in \dot{H}^{-\frac{1}{2}} \times \dot{H}^{\frac{3}{2}}$ due to the invalidity of $\dot{H}^{\frac{3}{2}} \hookrightarrow L^\infty$. Thus a relatively smaller initial data space: the hybrid Besov space $\dot{B}^{-\frac{1}{2}}_{2,1} \times (\dot{B}^{\frac{1}{2}}_{2,1} \cap \dot{B}^{\frac{3}{2}}_{2,1})$ was used in [9].

In the subcritical case, it is easy to check that the L^2 function $(1+|x|^2)^{-1}$ in 3-dimension neither belongs to $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ nor belongs to $\dot{B}^{-\frac{1}{2}}_{2,1}$. Thus $(u_0, \nabla w_0) \in L^2 \times H^1$ cannot be treated directly by applying the arguments for the critical case in [9]. Although we believe that the Chemin–Lerner space framework can be modified to handle the subcritical case, we shall employ a new approach to study (1.3) for initial data $(u_0, \nabla w_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ since the Fourier multiplier theory provides us with another option without using dyadic decompositions. Meanwhile, by recalling the well known weak solution theory for the heat equation, we observe that finding a unique solution

$$(u, \nabla w) \in \left(C\left([0,\infty); L^2\right) \cap L^2\left(0,\infty; \dot{H}^1\right)\right) \times C\left([0,\infty); H^1\right)$$

is also important. Following the similar arguments of [20], one can lower down the regular index s of the initial data space $H^s \times H^s$ for $(u_0, \nabla w_0)$. However, $L^2 \times H^1$ seems to be unreachable. Therefore our first goal is to apply the Fourier analysis tools to prove the well-posedness of (1.3) in $L^2 \times H^1$.

Our next goal is to establish the well-posedness of (1.3) with initial value $(u_0, \nabla w_0) \in H^2 \times H^1$. Precisely speaking, we will prove that the Cauchy problem (1.1)–(1.2) has a unique solution

$$\left(u - \bar{u}, \nabla\left((\mu - \bar{u})t + \ln v\right)\right) \in C\left([0,\infty); H^2\right) \times C\left([0,\infty); H^1\right)$$

provided that $(u_0, \nabla \ln v_0)$ is close enough to the equilibrium state $(\bar{u}, 0)$ in $H^2 \times H^1$, where $\bar{u}$ is defined in (1.6). From the system (1.3), it is natural to assume that the second derivatives of u and w exist almost everywhere, although certain higher derivatives might not exist.

In the four-dimensional (4D) case, scaling argument suggests that $L^2(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$ is the critical initial data space for $(u_0, \nabla w_0)$. Therefore, an interesting *problem* is whether (1.3) has a unique solution even locally in time for $(u_0, \nabla w_0) \in L^2(\mathbb{R}^4) \times H^1(\mathbb{R}^4)$.

We now define the mean value of u on $\mathbb{R}^3$ as

$$\bar{u} = \lim_{R\to\infty} \frac{1}{|B_R|} \int\limits_{B_R} u_0(x)\,dx, \tag{1.6}$$

where $B_R \subset \mathbb{R}^3$ is a space ball centered at the origin with radius $R > 0$ and u_0 is the initial cell density. Noticing that the mean value of u on $\mathbb{R}^3$ is a conserved quantity due to its conservative nature, hence $\bar{u}$ is well defined. Let $\bar{u} = 1$ for the sake of simplicity. By changing variables in (1.1): $p = u - \bar{u}$ and $h = (\mu - \bar{u})t + \ln v$, we get

$$\begin{cases} \partial_t p = \Delta p + \Delta h + \nabla \cdot (p \nabla h), \\ \partial_t h = p \end{cases}$$

where $(t, x) \in (0, \infty) \times \mathbb{R}^3$. It is easy to check that for any positive constant c, if (p, h) is a solution to the above system, then $(p, h + \ln c)$ is also a solution. Equivalently, if (u, v) is a solution to (1.1), then (u, cv) is also a solution. As a consequence, it is natural to choose ∇h as an unknown function. Let $\Lambda = \sqrt{-\Delta}$, $q = -\Lambda h$ and $G = \Lambda^{-1} \nabla \cdot (p \nabla \Lambda^{-1} q)$. Then we get

$$\begin{cases} \partial_t p = \Delta p + \Lambda q - \Lambda G, \\ \partial_t q = -\Lambda p \end{cases} \tag{1.7}$$

where $(t, x) \in (0, \infty) \times \mathbb{R}^3$. (1.7) is supplemented with initial data

$$(p, q)(0, x) = (p_0, q_0)(x), \quad p_0 = u_0 - \bar{u} \tag{1.8}$$

for $x \in \mathbb{R}^3$, where $p_0 = u_0 - \bar{u}$ and $q_0 = -\Lambda \ln v_0$.

Our proof of the global well-posedness of (1.7) with initial data $(p_0, q_0) \in H^k \times H^1$ $(k = 0, 2)$ is based on a combination of the Fourier transform and estimates of the eigenvalues of the corresponding characteristic matrix (see (3.1)–(3.13)). The different decay properties of the eigenvalues of the characteristic matrix enable us to take advantages of the smoothing properties of the *high* frequency piece[2] of p, i.e., $p \in L^1(0, \infty; \dot{H}_\psi^{7/4})$ instead of that of q since the high frequency piece of q does not have spatial smoothing effect (see (1.12)). The use of $L^1(0, \infty; \dot{H}_\psi^{7/4})$ is the *noval* point of this paper. The main difficulty is to estimate $\|p \nabla q\|_{L^1(0,\infty;L^2)}$, which forces us to use frequency decomposition and smoothing effects (see Lemma 3.2 below). Once $\|p \nabla q\|_{L^1(0,\infty;L^2)}$ being estimated, the desired result follows from a standard fixed point argument. As for the decay property of v in (1.1), we apply the limiting case of the Sobolev inequality in BMO (cf. [15]) to $v = ce^{(\bar{u}-\mu)t} e^{-\Lambda^{-1} q}$ to obtain both lower and upper bounds of its L^∞ norm which are stated in (1.20)–(1.21).

To prove our main results in Theorems 1.1 and 1.2, we shall apply the Fourier multiplier theory and the smoothing properties of the parabolic–hyperbolic coupled system (see (3.11)–(3.13)). In particular, from (3.6), (3.11) and $m_1(t, \xi)$ for $|\xi| > 2$ in (M), we observe that for $|\xi| > 4$,

$$m_1(t, \xi) = \frac{e^{-\frac{t(1+\Xi)|\xi|^2}{2}}}{\frac{2\Xi}{\Xi+1}} - \frac{e^{-\frac{2t}{1+\Xi}}}{\frac{\Xi(\Xi+1)|\xi|^2}{2}} \quad \text{with } \Xi = \sqrt{1 - \frac{4}{|\xi|^2}} \in \left(\frac{\sqrt{3}}{2}, 1 \right). \tag{1.9}$$

To explore the smoothing effect, we need to study the operator $\partial_t^k \partial^\alpha m_1(t, D)$ with symbol

$$\partial_t^k \xi^\alpha m_1(t, \xi) = -\frac{(1+\Xi)^{k+1}}{(-2)^{k+1} \Xi} |\xi|^{2k} \xi^\alpha e^{-\frac{t(1+\Xi)|\xi|^2}{2}} + \frac{(-2)^{k+1}}{\Xi(\Xi+1)^{k+1}} \xi^\alpha |\xi|^{-2} e^{-\frac{2t}{1+\Xi}}, \tag{1.10}$$

[2] Definitions of the low, medium and high frequency pieces and $\dot{H}_\psi^{7/4}$ are given by (1.15) and (1.17), respectively.

$\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $k \in \mathbb{N}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \le 2$. Indeed, for any $t > 0$, $|\xi| > 4$ and $p_0 \in L^2(\mathbb{R}^3)$, from (1.9)–(1.10) one gets

$$\left\| \partial_t^k \partial^\alpha m_1(t, D) p_0 \right\|_{L^2} \le \left(C_1(k) t^{-\frac{|\alpha|}{2} - k} + C_2(k) e^{-t} \right) \| p_0 \|_{L^2}. \tag{1.11}$$

However, by following the similar arguments, for any $t > 0$, $|\xi| > 4$ and $q_0 \in H^1(\mathbb{R}^3)$, one can only get from (3.6), (M) and (3.11) that

$$\left\| \partial_t^k m_2(t, D) q_0 \right\|_{H^1} \le C_3(k) \| q_0 \|_{H^1} \tag{1.12}$$

where no smoothing effect exists for spatial variable. For the low frequency piece, i.e. $|\xi| \le C$ (C is some fixed positive constant), since $m_1(t, D)$ and $m_2(t, D)(-\Delta)^{-\frac{1}{2}}$ behave similarly to $e^{t\Delta}$ in the Sobolev space settings, we omit the detail smoothing argument. It is worth pointing out that the linear part of (1.7) is also known as the weak dissipative structure, see for instance [4,7,16] and the references therein.

Before stating our main results, let us define the partition of unit. Let us briefly explain how it may be built in $\mathbb{R}^3$. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwarz class and (η, φ, ψ) be three smooth radially symmetric functions with range in $[0, 1]$ such that

$$\operatorname{supp} \psi \subset \left\{ \xi \in \mathbb{R}^3;\ |\xi| > 2^4 \right\}, \qquad \operatorname{supp} \varphi \subset \left\{ \xi \in \mathbb{R}^3;\ 1 < |\xi| < 2^5 \right\}, \tag{1.13}$$

$$\operatorname{supp} \eta \subset \left\{ \xi \in \mathbb{R}^3;\ |\xi| < 2 \right\}, \qquad \eta(\xi) + \varphi(\xi) + \psi(\xi) = 1, \quad \forall \xi \in \mathbb{R}^3. \tag{1.14}$$

For $f \in \mathcal{S}'(\mathbb{R}^3)$, we define the low, medium and high frequency operators as follows[3]:

$$f^l = \eta(D) f, \qquad f^m = \varphi(D) f, \qquad f^h = \psi(D) f, \qquad \eta(D)\psi(D) f \equiv 0 \tag{1.15}$$

with $\eta(\xi), \varphi(\xi)$ and $\psi(\xi)$ being the symbols of $\eta(D), \varphi(D)$ and $\psi(D)$, respectively.

Throughout this paper, $\mathcal{F} f$ and $\widehat{f}$ stand for Fourier transform of f with respect to space variables and $\mathcal{F}^{-1}$ stands for the corresponding inverse Fourier transform. For any $s \ge 0$ and any function f, we shall define the fractional Riesz potential Λ^s and Bessel potential $\langle \Lambda \rangle^s := (1 - \Delta)^{\frac{s}{2}}$ by

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi) \quad \text{and} \quad \widehat{\langle \Lambda \rangle^s f}(\xi) = \langle \xi \rangle^s \widehat{f}(\xi) = \left(1 + |\xi|^2\right)^{\frac{s}{2}} \widehat{f}(\xi) \tag{1.16}$$

respectively.

Recall from (1.4)–(1.5) and the definitions of Λ and $\langle \Lambda \rangle$ that for any $s > 0$, it holds that $H^s = \dot{H}^s \cap L^2$. For $s \in \mathbb{R}$, we define

$$\dot{H}^s_\psi = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^3\right);\ \| f \|_{\dot{H}^s_\psi} = \left\| \Lambda^s \psi(D) f \right\|_{L^2} = \left\| \Lambda^s f^h \right\|_{L^2} < \infty \right\} \tag{1.17}$$

where $\dot{H}^s_\psi$ itself is not a Banach space since by using (1.15) one can prove that for any $g \in \mathcal{S}(\mathbb{R}^3)$ satisfying $\operatorname{supp} \widehat{g} \subset \{\xi \in \mathbb{R}^3;\ |\xi| < 2^4\}$ and $f \in \dot{H}^s_\psi$, it holds that $\| f \|_{\dot{H}^s_\psi} = \| f + g \|_{\dot{H}^s_\psi}$. Hence

[3] $f^l = \eta(D) f = \mathcal{F}^{-1}(\eta(\xi) \widehat{f}(\xi))$ and similar conventions are used in this paper.

we need to introduce another Banach space Z such that $Z \cap \dot{H}^s_\psi$ is a Banach space. We denote $C([0,\infty); X(\mathbb{R}^3))$ the Banach space with norm $\|\cdot\|_{L^\infty_t X}$, where we omit the space domain for the sake of simplicity if there is no confusion. $A \lesssim B$ represents $A \le CB$ for some positive constant C which dependence on various parameters is clear. $A \sim B$ iff $A \lesssim B \lesssim A$. For any $1 \le \rho, r \le \infty$, we denote $L^\rho(0,\infty)$ and $L^\rho((0,\infty); L^r)$ by L^ρ_t and $L^\rho_t L^r$, respectively.

We state our main results as follows.

Theorem 1.1. *There exist $C, \varepsilon_0 > 0$ so that if $\|(p_0, q_0)\|_{L^2 \times H^1} \le \varepsilon_0$, then Cauchy problem (1.7)–(1.8) has a unique global solution $(p,q) \in C([0,\infty); L^2) \times C([0,\infty); H^1)$ satisfying*

$$\big\|(p,q)\big\|_{L^\infty_t L^2 \times L^\infty_t H^1} + \big\|(\nabla p, \nabla q)\big\|_{L^2_t L^2 \times L^2_t L^2} + \|p\|_{L^1_t \dot{H}^{7/4}_\psi} \le C\varepsilon_0$$

for all $t > 0$.

Theorem 1.2. *There exist constants $C, \varepsilon_0 > 0$ so that if $\|(p_0, q_0)\|_{H^2 \times H^1} \le \varepsilon_0$, then Cauchy problem (1.7)–(1.8) has a unique global solution $(p,q) \in C([0,\infty); H^2) \times C([0,\infty); H^1)$. Moreover, for any $t > 0$,*

$$\big\|(p,q)\big\|_{L^\infty_t H^2 \times L^\infty_t H^1} + \sup_{t>0}(1+t)^{\frac{1}{2}}\big\|(\nabla p, \nabla q)\big\|_{L^2 \times L^2} + \sup_{t>0}(1+t)^{\frac{7}{8}}\big\|\Lambda^{\frac{7}{4}} p\big\|_{L^2} \le C\varepsilon_0.$$

Recall that if (u, v) solves (1.1), then for any positive constant c, (u, cv) also solves (1.1). Hence from the unique solution (p,q) of (1.7), we get a sequence of solutions (u, cv) such that $v = ce^{(\bar{u}-\mu)t} e^{-\Lambda^{-1} q}$. From embedding theorems[4] $\dot{H}^{\frac{5}{4}} \hookrightarrow L^{12}$, $\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow BMO^{-1}$ and Lemma 2.5 below, we get $v = ce^{(\bar{u}-\mu)t} e^{-\Lambda^{-1} q}$ and

$$\begin{aligned}
\big\|\Lambda^{-1} q\big\|_{L^\infty} &\le C\big(1 + \big\|\Lambda^{-1} q\big\|_{BMO}\big(1 + \max\big\{0, \ln\big\|\Lambda^{-1} q\big\|_{W^{\frac{3}{4},12}}\big\}\big)\big) \\
&\le C\big(1 + \|q\|_{BMO^{-1}}\big(1 + \max\big\{0, \ln\big(\big\|\Lambda^{-1} q\big\|_{L^{12}} + \big\|\Lambda^{-\frac{1}{4}} q\big\|_{L^{12}}\big)\big\}\big)\big) \\
&\le C\big(1 + \|q\|_{\dot{H}^{\frac{1}{2}}}\big(1 + \max\big\{0, \ln\big(\|q\|_{\dot{H}^{\frac{1}{4}}} + \|q\|_{\dot{H}^1}\big)\big\}\big)\big) \\
&\le C\big(1 + \|\nabla q\|_{L^2}^{\frac{1}{2}} \|q\|_{L^2}^{\frac{1}{2}}\big(1 + \max\big\{0, \ln\|q\|_{H^1}\big\}\big)\big).
\end{aligned} \tag{1.18}$$

In Theorem 1.2, we choose ε_0 such that $C\varepsilon_0 \le 1$. Then from (1.18), we obtain that

$$\big\|\Lambda^{-1} q\big\|_{L^\infty} \le C\big(1 + \|\nabla q\|_{L^2}^{\frac{1}{2}} \|q\|_{L^2}^{\frac{1}{2}}\big) \le C\big(1 + \|\nabla q\|_{L^2}^{\frac{1}{2}}\big). \tag{1.19}$$

Applying (1.19) to $v = ce^{(\bar{u}-\mu)t} e^{-\Lambda^{-1} q}$, we get

$$\frac{1}{c}\|v\|_{L^\infty} = e^{(\bar{u}-\mu)t}\big\|e^{-\Lambda^{-1} q}\big\|_{L^\infty} \le e^{(\bar{u}-\mu)t} e^{\|\Lambda^{-1} q\|_{L^\infty}} \le e^{(\bar{u}-\mu)t} e^{C(1+\|\nabla q\|_{L^2}^{\frac{1}{2}})}, \tag{1.20}$$

$$\frac{1}{c}\|v\|_{L^\infty} = e^{(\bar{u}-\mu)t}\big\|e^{-\Lambda^{-1} q}\big\|_{L^\infty} \ge e^{(\bar{u}-\mu)t} e^{-\|\Lambda^{-1} q\|_{L^\infty}} \ge e^{(\bar{u}-\mu)t} e^{-C(1+\|\nabla q\|_{L^2}^{\frac{1}{2}})}. \tag{1.21}$$

[4] We refer the readers to [29,14] to see the definitions of BMO and BMO^{-1} as well as $L^n \hookrightarrow BMO^{-1}$.

From Theorem 1.2 and (1.20)–(1.21), we have the following results on the asymptotic behavior of solutions. In particular, if the mean value of the initial cell density is smaller than a constant μ, then the cell density approaches to its initial mean and the chemical concentration decays exponentially to zero as t goes to infinity.

Corollary 1.3. *If $(u_0 - \bar{u}, \nabla \ln v_0) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and there exists constant $\varepsilon_0 > 0$ such that $\|(u_0 - \bar{u}, \nabla \ln v_0)\|_{H^2 \times H^1} \le \varepsilon_0$, then Cauchy problem (1.1)–(1.2) has a global solution (u, v) satisfying that for all $t > 0$*

$$\big\|(u - \bar{u}, \nabla \ln v)\big\|_{L_t^\infty H^2 \times L_t^\infty H^1} \lesssim \varepsilon_0,$$
$$(u - \bar{u}, \nabla \ln v) \in C\big([0, \infty); H^2\big) \times C\big([0, \infty); H^1\big)$$

and

$$\sup_{t>0}(1+t)^{\frac{1}{2}}\big\|(\nabla u, \Delta \ln v)\big\|_{L^2 \times L^2} + \sup_{t>0}(1+t)^{\frac{7}{8}}\big\|\Lambda^{\frac{7}{4}} u\big\|_{L^2} \lesssim \varepsilon_0.$$

Moreover

$$\|u - \bar{u}\|_{L^\infty} \lesssim (1+t)^{-\frac{1}{2}} \quad \text{as } t \to \infty; \qquad \|v\|_{L^\infty} \sim e^{(\bar{u}-\mu)t} \quad \text{as } t \to \infty. \tag{1.22}$$

In particular, if $\mu > \bar{u}$, then

$$\big\|(u - \bar{u}, v)\big\|_{L^\infty} \to (0, 0)$$

as $t \to \infty$.

Plan of the paper: In Section 2 we prove several preliminaries lemmas, and in Section 3 we prove Theorems 1.1 and 1.2 and Corollary 1.3.

2. Preliminary lemmas

In this section, we always assume that dimension $n = 3$. We now list several known lemmas which will be used to prove the well-posedness of (1.7)–(1.8).

The first lemma is concerned with functions whose Fourier transforms are supported in low, medium and high frequency areas, respectively. We note that the first two results in the first lemma are the well-known Bernstein's inequalities (cf. [17, Proposition 3.2, p. 24] and [1, Lemma 2.1, p. 52]) and the last one is a direct consequence of the Sobolev embedding theorem.

Lemma 2.1. *If $(s, a, b) \in [0, \infty) \times [1, \infty]^2$, $a \le b$ and $f(x) \in L^a$, then for any two positive constants c_1 and c_2, there exists positive constant c such that*

$$\operatorname{supp} \widehat{f} \subset \big\{\xi \in \mathbb{R}^3;\ |\xi| \le c_2\big\}, \qquad \big\|\Lambda^s f\big\|_{L^b} \le c c_2^{s+n(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}, \tag{2.1}$$

$$\operatorname{supp} \widehat{f} \subset \big\{\xi \in \mathbb{R}^3;\ c_1 < |\xi| < c_2\big\}, \qquad \frac{c}{\kappa^{c_1^s}}\|f\|_{L^a} \le \big\|\Lambda^s f\big\|_{L^a} \le c\kappa^{c_2^s}\|f\|_{L^a}, \tag{2.2}$$

$$\operatorname{supp}\widehat{f}\subset\{\xi\in\mathbb{R}^3;\ |\xi|\ge c_1\},\qquad \|f\|_{L^a}\le\|\langle\Lambda\rangle^s f\|_{L^a}=\|f\|_{W^{s,p}},\tag{2.3}$$

where $\kappa=\ln\frac{c_2}{c_1}$ *and* $W^{s,p}$ *is the fractional Sobolev space.*

Applying Lemma 2.1 for $2=a\le b\le\infty$ and $s\ge 0$ to $\eta(D)f$ and $\psi(D)f$, we get

$$\|\eta(D)f\|_{L^b}\lesssim\|f\|_{L^2}\quad\text{and}\quad\|\psi(D)f\|_{L^2}\lesssim\|\Lambda^s f\|_{L^2}.$$

From (2.1)–(2.2), we have the following lemma concerning the L^2 Fourier multiplier.

Lemma 2.2. *If* $r\in[1,\infty]$, $v\in L^2$, $m(t,\xi)\in L_t^rL_\xi^\infty$ [5] *and* $m(t,D)v=\mathcal{F}^{-1}m(t,\xi)\widehat{v}(\xi)$, *then we get*

$$\|m(t,D)v\|_{L_t^rL^2}\le\|m\|_{L_t^rL_\xi^\infty}\|v\|_{L^2};\tag{2.4}$$

if $r\in[2,\infty]$, $v\in L^2$, $m(t,\xi)|\xi|^s\in L_t^rL_\xi^\infty$ *and* $m(t,D)v=\mathcal{F}^{-1}m(t,\xi)\widehat{v}(\xi)$, *then we get*

$$\|m(t,D)v\|_{L_t^r\dot{H}^s}\le\sup_{\xi\in\mathbb{R}^3}\left(|\xi|^s\|m(\cdot,\xi)\|_{L_t^r}\right)\|v\|_{L^2}.\tag{2.5}$$

Proof. The proof of (2.4) follows from the classical Fourier multiplier theory and for completeness, we give the proof as follows

$$\|m(t,D)v\|_{L_t^rL^2}=\|m(t,\cdot)\widehat{v}(\cdot)\|_{L_t^rL_\xi^2}\le\big\|\|m(t,\cdot)\|_{L_\xi^\infty}\big\|_{L_t^r}\|\widehat{v}\|_{L_\xi^2}\le\|m\|_{L_t^rL_\xi^\infty}\|v\|_{L^2}.$$

In order to prove (2.5), we need to use Plancherel's equality, Minkowski's inequality, Hölder's inequality and Plancherel's equality again, i.e.,

$$\begin{aligned}\|m(t,D)v\|_{L_t^r\dot{H}^s}&=\|m(t,\cdot)|\cdot|^s\widehat{v}(\cdot)\|_{L_t^rL_\xi^2}\lesssim\|m(t,\cdot)|\cdot|^s\widehat{v}(\cdot)\|_{L_\xi^2L_t^r}\\&\le\sup_{\xi\in\mathbb{R}^3}\left(\|m(\cdot,\xi)\|_{L_t^r}|\xi|^s\right)\|\widehat{v}\|_{L_\xi^2}.\end{aligned}$$

Hence we finish the proof. □

The skill we used in proving Lemma 2.2 will be used repeatedly in the rest of the paper.

Lemma 2.3. *If* $|m(t,\xi)|\le c_1(\frac{e^{-ct}}{1+|\xi|^2}+e^{-ct|\xi|^2})$ *and* $|n(t,\xi)|\le c_1e^{-ct}$ *for some positive constants* c *and* c_1, $2\le\rho\le\infty$, $1\le r\le\rho_1\le\infty$ *and* $(u,v)\in L_t^2L^2\times L_t^rL^2$, *then we get*

[5] $\|m\|_{L_t^rL_\xi^\nu}^r=\int_0^\infty\|m(t,\cdot)\|_{L^\nu(\mathbb{R}^3)}^r\,dt$ and similar conventions are used.

$$\left\| \int_0^t m(t-\tau, D)u(\tau)d\tau \right\|_{L_t^\rho \dot{H}^{1+\frac{2}{\rho}}} \lesssim \|u\|_{L_t^2 L^2}, \tag{2.6}$$

$$\left\| \int_0^t n(t-\tau, D)u(\tau)d\tau \right\|_{L_t^{\rho_1} L^2} \lesssim \|u\|_{L_t^r L^2}. \tag{2.7}$$

Proof. By applying the Plancherel equality, Lemma 2.2 with $\|m(t,\cdot)\|_{L_\xi^\infty} \le c_1 e^{-ct}$ and the integrability of e^{-ct}, we see that the proof of (2.7) is quite straightforward. Thus it suffices to prove (2.6). Since $1+\frac{2}{\rho} \in [0,2]$, by making use of the definition of the Fourier transformation and the Fubini theorem, we get

$$\begin{aligned}
\left\| \int_0^t m(t-\tau, D)u(\tau)d\tau \right\|_{L_t^\rho \dot{H}^{1+\frac{2}{\rho}}} &= \left\| \int_0^t m(t-\tau, \xi)|\xi|^{1+\frac{2}{\rho}} \widehat{u}(\tau)d\tau \right\|_{L_t^\rho L_\xi^2} \\
&\lesssim \left\| \int_0^t m(t-\tau, \xi)|\xi|^{1+\frac{2}{\rho}} \widehat{u}(\tau)d\tau \right\|_{L_\xi^2 L_t^\rho} \\
&\lesssim \left\| \int_0^t \left(e^{-c(t-\tau)|\xi|^2}|\xi|^{1+\frac{2}{\rho}} + e^{-c(t-\tau)}\right)\left|\widehat{u}(\tau)\right| d\tau \right\|_{L_\xi^2 L_t^\rho} \\
&\lesssim \big\| \|\widehat{u}\|_{L_t^2} \big\|_{L_\xi^2} \sim \|\widehat{u}\|_{L_t^2 L_\xi^2} \lesssim \|u\|_{L_t^2 L^2},
\end{aligned}$$

where in the second, fourth and fifth inequalities we have used Minkowski, Young's inequality, Fubini's theorem and Plancherel's equality. Hence we finish the proof. □

The next lemma is about the Picard contraction argument (cf. [2]) which will be used to prove the well-posedness of (1.7) with initial data $(p_0, q_0) = (x_{10}, x_{20}) \in X_{10} \times X_{20} = L^2 \times H^1$ or $H^2 \times H^1$.

Lemma 2.4. *Let $(X_{10} \times X_{20}, \|\cdot\|_{X_{10}} + \|\cdot\|_{X_{20}})$ and $(X_1 \times X_2, \|\cdot\|_{X_1} + \|\cdot\|_{X_2})$ be abstract Banach product spaces. Let $L_1 : X_{10} \times X_{20} \to X_1$, $L_2 : X_{10} \times X_{20} \to X_2$, $B_1 : X_1 \times X_2 \to X_1$ and $B_2 : X_1 \times X_2 \to X_2$ be two linear and two bilinear operators such that if for any $(x_{10}, x_{20}) \in X_{10} \times X_{20}$, $(x_1, x_2) \in X_1 \times X_2$, $c > 0$ and $i = 1, 2$,*

$$\big\|L_i(x_{10}, x_{20})\big\|_{X_i} \le c\big(\|x_{10}\|_{X_{10}} + \|x_{20}\|_{X_{20}}\big) \quad \text{and} \quad \big\|B_i(x_1, x_2)\big\|_{X_i} \le c\|x_1\|_{X_1}\|x_2\|_{X_2},$$

then for any $(x_{10}, x_{20}) \in X_{10} \times X_{20}$ with $\|(x_{10}, x_{20})\|_{X_{10}\times X_{20}} < \frac{1}{16c}$, the following system

$$(x_1, x_2) = \big(L_1(x_{10}, x_{20}), L_2(x_{10}, x_{20})\big) + \big(B_1(x_1, x_2), B_2(x_1, x_2)\big)$$

has a solution (x_1, x_2) in $X_1 \times X_2$. In particular, the solution satisfying

$$\left\|(x_1, x_2)\right\|_{X_1 \times X_2} \leq 4\left\|(x_{10}, x_{20})\right\|_{X_{10} \times X_{20}}$$

is the only one such that $\|(x_1, x_2)\|_{X_1 \times X_2} < \frac{1}{4c}$.

The last lemma is the limiting case of the Sobolev inequality in BMO, see [15].

Lemma 2.5. *If* $n = 3$ *and* $s > \frac{1}{4}$, *then there exists a constant* C *depending on* s *so that*

$$\|f\|_{L^\infty} \leq C\left(1 + \|f\|_{BMO}\left(1 + \max\left\{0, \ln \|f\|_{W^{s,12}}\right\}\right)\right)$$

for all $f \in W^{s,12}$ *and* $\|f\|_{W^{s,12}} = \|\langle \Lambda \rangle^s f\|_{L^{12}}$.

3. Proof of the main results

In this section, we shall use the Fourier analysis tools to study the well-posedness of (1.7) with initial data in the Sobolev spaces. For the sake of simplicity, we also assume that $n = 3$ throughout this section.

3.1. *Linearization of (1.7) and the corresponding integral equations*

In this subsection, we first study the linearized system of (1.7) around $(0, 0)$

$$\frac{d}{dt}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\Lambda^2 & \Lambda \\ -\Lambda & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}. \tag{3.1}$$

Taking Fourier transform of (3.1) with respect to the space variable yields

$$\frac{d}{dt}\begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} = L(\xi)\begin{pmatrix} \widehat{p} \\ \widehat{q} \end{pmatrix} \quad \text{with } L(\xi) = \begin{pmatrix} -|\xi|^2 & |\xi| \\ -|\xi| & 0 \end{pmatrix}.$$

The characteristic polynomial of $L(\xi)$ is $X^2 + |\xi|^2 X + |\xi|^2$. According to the size of $|\xi|$, we have the following three subcases:

- If $|\xi| > 2$, then the characteristic polynomial possesses two distinct *real* roots: $\lambda_+ = \frac{|\xi|^2}{2}(-1 + \Xi)$ and $\lambda_- = \frac{|\xi|^2}{2}(-1 - \Xi)$ with $\Xi := \sqrt{1 - \frac{4}{|\xi|^2}}$. Since $\lambda_+ \neq \lambda_-$, the matrix $L(\xi)$ is diagonalizable. After computing the associated eigenspaces, we find that

$$\widehat{p} = \left(\frac{e^{t\lambda_-} + e^{t\lambda_+}}{2} + \frac{e^{t\lambda_-} - e^{t\lambda_+}}{2\Xi}\right)\widehat{p}_0 + \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\Xi}\frac{\widehat{q}_0}{|\xi|}, \tag{3.2}$$

$$\widehat{q} = \frac{e^{t\lambda_-} - e^{t\lambda_+}}{\Xi}\frac{\widehat{p}_0}{|\xi|} + \left(\frac{e^{t\lambda_-} + e^{t\lambda_+}}{2} + \frac{e^{t\lambda_+} - e^{t\lambda_-}}{2\Xi}\right)\widehat{q}_0, \tag{3.3}$$

where, for simplicity, we denote $\frac{e^{t\lambda_+} + e^{t\lambda_-}}{2}$ and $\frac{e^{t\lambda_+} - e^{t\lambda_-}}{2\Xi}$ by $\Omega_{1,t}(\xi)$ and $\Omega_{2,t}(\xi)$, respectively. Moreover, if there is no confusion, we will denote $\Omega_{1,t}(\xi)$ and $\Omega_{2,t}(\xi)$ by $\Omega_{1,t}$ and $\Omega_{2,t}$, respectively.

• If $|\xi| < 2$, then the characteristic polynomial has two distinct *complex* roots: $\lambda_+ = -\frac{|\xi|^2}{2} - i\frac{\Theta|\xi|^2}{2}$ and $\lambda_- = -\frac{|\xi|^2}{2} + i\frac{\Theta|\xi|^2}{2}$ with $\Theta := \sqrt{-1 + \frac{4}{|\xi|^2}}$. Noticing that $\lambda_+ \neq \lambda_-$, hence the matrix $L(\xi)$ is also diagonalizable. After computing the associated eigenspaces, we get

$$\widehat{p} = \left(\frac{e^{t\lambda_-} + e^{t\lambda_+}}{2} + \frac{e^{t\lambda_-} - e^{t\lambda_+}}{-2i\Theta}\right)\widehat{p}_0 + \frac{e^{t\lambda_+} - e^{t\lambda_-}}{-i\Theta}\frac{\widehat{q}_0}{|\xi|}, \tag{3.4}$$

$$\widehat{q} = \frac{e^{t\lambda_-} - e^{t\lambda_+}}{-i\Theta}\frac{\widehat{p}_0}{|\xi|} + \left(\frac{e^{t\lambda_-} + e^{t\lambda_+}}{2} + \frac{e^{t\lambda_+} - e^{t\lambda_-}}{-2i\Theta}\right)\widehat{q}_0, \tag{3.5}$$

where, for simplicity, we denote $\frac{e^{t\lambda_+}+e^{t\lambda_-}}{2}$ and $\frac{e^{t\lambda_+}-e^{t\lambda_-}}{-2i\Theta}$ by $\Omega_{3,t}$ and $\Omega_{4,t}$, respectively.
• If $|\xi| = 2$, then $L(\xi)$ is not diagonalizable. However, this case can be defined via $\lim_{|\xi|\to 2^+}$ and $\lim_{|\xi|\to 2^-}$ since the two limits exist and coincide.

We divide the analysis of the multipliers in (3.2)–(3.5) into the following five subcases:

If $|\xi| > 4$, then we obtain that $\frac{\sqrt{3}}{2} < \Xi < 1$, $\lambda_+ = -\frac{2}{1+\Xi}$, $\lambda_- = -\frac{(1+\Xi)|\xi|^2}{2}$, $\Omega_{2,t} = \frac{e^{t\lambda_+}-e^{t\lambda_-}}{2\Xi} = \frac{e^{-\frac{2t}{1+\Xi}}(1-e^{-t|\xi|^2\Xi})}{2\Xi}$ and $\Omega_{1,t} - \Omega_{2,t} = \frac{e^{t\lambda_-}+e^{t\lambda_+}}{2} + \frac{e^{t\lambda_-}-e^{t\lambda_+}}{2\Xi} = \frac{e^{-\frac{t(1+\Xi)|\xi|^2}{2}}}{\frac{2\Xi}{\Xi+1}} - \frac{e^{-\frac{2t}{1+\Xi}}}{\frac{\Xi(\Xi+1)|\xi|^2}{2}}$ which gives

$$|\Omega_{1,t} - \Omega_{2,t}| \le 2e^{-\frac{t|\xi|^2}{2}} + 3e^{-t}\frac{1}{1+|\xi|^2} \quad \text{and} \quad |\Omega_{2,t}| \le e^{-t}. \tag{3.6}$$

If $2 < |\xi| \le 4$, then we have $0 < \Xi \le \frac{\sqrt{3}}{2}$, $\lambda_+ = -\frac{2}{1+\Xi}$, $\lambda_- = -\frac{(1+\Xi)|\xi|^2}{2}$ and $\Omega_{1,t} = \frac{e^{t\lambda_-}+e^{t\lambda_+}}{2} = \frac{e^{-\frac{t(1+\Xi)|\xi|^2}{2}}+e^{-\frac{2t}{1+\Xi}}}{2}$. Applying $1 - e^{-|x|} \le |x|$ to $\Omega_{2,t}$ and noticing that $4 < |\xi|^2 \le 16$, it holds

$$|\Omega_{1,t}| \le e^{-t} \quad \text{and} \quad |\Omega_{2,t}| \le 16e^{-\frac{t}{2}}. \tag{3.7}$$

If $1 \le |\xi| < 2$, then we obtain that $0 < \Theta|\xi| \le \sqrt{3}$, $\lambda_\pm = -\frac{|\xi|^2}{2} \mp i\frac{\Theta|\xi|^2}{2}$, $\Omega_{3,t} = \frac{e^{t\lambda_+}+e^{t\lambda_-}}{2} = e^{-\frac{t|\xi|^2}{2}}\cos\frac{\Theta|\xi|^2 t}{2}$ and $\frac{\Omega_{4,t}}{|\xi|} = \frac{e^{t\lambda_+}-e^{t\lambda_-}}{-2i\Theta|\xi|} = \frac{1}{2}e^{-\frac{t|\xi|^2}{2}}\frac{\sin\frac{\Theta|\xi|^2 t}{2}}{\frac{\Theta|\xi|}{2}} = \frac{1}{2}e^{-\frac{t|\xi|^2}{2}}\frac{\sin\frac{\Theta|\xi|^2 t}{2}}{\frac{\Theta|\xi|^2 t}{2}}t|\xi|$. Applying $|\sin x| \le |x|$, $|\cos x| \le 1$ to $\Omega_{3,t}$ and $\Omega_{4,t}$, we get

$$\frac{|\Omega_{4,t}|}{|\xi|} \le 4e^{-\frac{t}{4}}, \qquad |\Omega_{4,t}| \le 8e^{-\frac{t}{4}} \quad \text{and} \quad |\Omega_{3,t}| \le e^{-\frac{t}{2}}. \tag{3.8}$$

If $|\xi| < 1$, then we can prove that $\sqrt{3} < \Theta|\xi| < 2$, $\lambda_\pm = -\frac{|\xi|^2}{2} \mp i\frac{\Theta|\xi|^2}{2}$,

$$\frac{|\Omega_{4,t}|}{|\xi|} \le e^{-\frac{t|\xi|^2}{2}}, \qquad |\Omega_{4,t}| \le 4e^{-\frac{t|\xi|^2}{4}} \quad \text{and} \quad |\Omega_{3,t}| \le 2e^{-\frac{t|\xi|^2}{2}}. \tag{3.9}$$

If $|\xi| \to 2$, then we get $\lim_{|\xi|\to 2^+} \Xi = \lim_{|\xi|\to 2^-} \Theta = 0$, $\lim_{|\xi|\to 2} \lambda_+ = \lim_{|\xi|\to 2} \lambda_- = -2$ and

$$\lim_{|\xi|\to 2^+} \Omega_{1,t} = \lim_{|\xi|\to 2^-} \Omega_{3,t} = e^{-2t}, \qquad \lim_{|\xi|\to 2^+} \Omega_{2,t} = \lim_{|\xi|\to 2^-} \Omega_{4,t} = 2te^{-2t}. \tag{3.10}$$

For simplicity, we define the following two multipliers:

$$m_1(t,\xi) = \begin{cases} \Omega_{1,t} - \Omega_{2,t} & \text{if } |\xi| > 2, \\ e^{-2t} - 2te^{-2t} & \text{if } |\xi| = 2, \\ \Omega_{3,t} - \Omega_{4,t} & \text{if } |\xi| < 2, \end{cases} \qquad m_2(t,\xi) = \begin{cases} \Omega_{2,t} & \text{if } |\xi| > 2, \\ 2te^{-2t} & \text{if } |\xi| = 2, \\ \Omega_{4,t} & \text{if } |\xi| < 2. \end{cases} \tag{M}$$

Applying (3.6)–(3.10) to $m_1(t,\xi)$ and $m_2(t,\xi)$, we observe that $m_1(t,\xi)$ and $m_2(t,\xi)$ are not only radial but also continuous with respect to frequency variable ξ. Moreover, there exist constants c and c_1 such that if $|\xi| > 2^4$, then we get

$$\left|m_1(t,\xi)\right| \le c_1\left(e^{-ct|\xi|^2} + e^{-ct}\frac{1}{1+|\xi|^2}\right) \quad \text{and} \quad \left|m_2(t,\xi)\right| \le c_1 e^{-ct}; \tag{3.11}$$

if $1 < |\xi| < 2^5$, then we get

$$\left|m_1(t,\xi)\right| + \left|m_2(t,\xi)\right| \le c_1 e^{-ct}; \tag{3.12}$$

else if $|\xi| < 2$, then we get

$$\left|m_1(t,\xi)\right| + \left|m_2(t,\xi)\right| + \frac{|m_2(t,\xi)|}{|\xi|} \le c_1 e^{-ct|\xi|^2}. \tag{3.13}$$

Integral equations. We rewrite (1.7) with initial data (p_0, q_0) into the following equivalent integral equations

$$p = m_1(t,D)p_0 + 2m_2(t,D)\Lambda^{-1}q_0 - \int_0^t m_1(t-\tau,D)\Lambda G(\tau)\,d\tau, \tag{3.14}$$

$$q = -2m_2(t,D)\Lambda^{-1}p_0 + \big(m_1(t,D) + 2m_2(t,D)\big)q_0 - 2\int_0^t m_2(t-\tau,D)G(\tau)\,d\tau, \tag{3.15}$$

where $m_1(t,D)$ and $m_2(t,D)$ are symbols of $m_1(t,\xi)$ and $m_2(t,\xi)$, respectively. From (3.14) and (3.15), for any $(p_0,q_0) \in L^2 \times H^1$, we define $\mathfrak{F}$ such that

$$\mathfrak{F}(p,q) = \big(\mathfrak{F}_1(p,q), \mathfrak{F}_2(p,q)\big) = \big(\text{“r.h.s.” of (3.14)}, \text{“r.h.s.” of (3.15)}\big), \tag{3.16}$$

where “r.h.s.” stands for “right hand side”.

The proof of Theorem 1.2 is similar but simpler than that of Theorem 1.1, thus we prove Theorem 1.1 first.

3.2. *Proof of Theorem 1.1*

In this subsection, we first prove several *a priori* estimates including the crucial bilinear estimates. We define the corresponding resolution spaces as follows

$$X\times Y=\left\{(p,q)\in C\big([0,\infty);L^2\big)\times C\big([0,\infty);H^1\big)\text{ and }\|p\|_X+\|q\|_Y<\infty\right\}\tag{3.17}$$

where $\|p\|_X:=\|p\|_{L_t^\infty L^2}+\|p\|_{L_t^2\dot H^1}+\|p\|_{L_t^1\dot H_\psi^{\frac{7}{4}}}$ and $\|q\|_Y:=\|q\|_{L_t^\infty H^1}+\|q\|_{L_t^2\dot H^1}$.

In what follows, we prove several key estimates.

Proposition 3.1. *Let* (p,q) *be a solution to* (1.7) *with* $(p_0,q_0)\in L^2\times H^1$ *and* $\mathfrak{F}$ *and* $\mathfrak{F}_1$ *be defined as in* (3.16). *Then there hold*

$$\big\|\mathfrak{F}(p,q)\big\|_{L_t^\infty L^2\times L_t^\infty H^1}\lesssim\big\|(p_0,q_0)\big\|_{L^2\times H^1}+\|G\|_{L_t^2L^2}+\|G\|_{L_t^1\dot H^1},\tag{3.18}$$

$$\big\|\mathfrak{F}(p,q)\big\|_{L_t^2\dot H^1\times L_t^2\dot H^1}\lesssim\big\|(p_0,q_0)\big\|_{L^2\times H^1}+\|G\|_{L_t^2L^2}+\|G\|_{L_t^1\dot H^1},\tag{3.19}$$

$$\big\|\mathfrak{F}_1(p,q)\big\|_{L_t^1\dot H_\psi^{7/4}}\lesssim\big\|(p_0,q_0)\big\|_{L^2\times H^1}+\|G\|_{L_t^1\dot H^1}.\tag{3.20}$$

Proof. In order to prove (3.18)–(3.20), we have to establish several estimates whose proof will be divided into three parts.

Part I. *Estimate of* $\|\mathfrak{F}(p,q)\|_{L_t^\infty L^2\times L_t^\infty H^1}$. At first, we derive the estimate for $\mathfrak{F}_1(p,q)$. Since any L_ξ^∞ function $m(\xi)$ is an H^s (or $\dot H^s$) multiplier, we get from (3.11)–(3.13) that $m_1(t,\xi)$, $\frac{2m_2(t,\xi)}{|\xi|}\in L_t^\infty L_\xi^\infty$. Hence by applying (2.4) with $(r,s)=(\infty,0)$ to $m_1(t,D)p_0+2m_2(t,D)\Lambda^{-1}q_0$, we get

$$\big\|m_1(t,D)p_0+2m_2(t,D)\Lambda^{-1}q_0\big\|_{L_t^\infty L^2}\lesssim\|p_0\|_{L^2}+\|q_0\|_{L^2}.\tag{3.21}$$

From (3.11)–(3.13) and (3.21), we get $m_1(t,\xi)+2m_2(t,\xi)$, $\frac{\langle\xi\rangle m_2(t,\xi)}{|\xi|}\in L_t^\infty L_\xi^\infty$. Applying (2.4) for $(r,s)=(\infty,1)$ to $m_2(t,D)\Lambda^{-1}p_0+(m_1(t,D)+2m_2(t,D))q_0$ gives

$$\begin{aligned}&\big\|-2m_2(t,D)\Lambda^{-1}p_0+\big(m_1(t,D)+2m_2(t,D)\big)q_0\big\|_{L_t^\infty H^1}\\&\quad\lesssim\big\|2m_2(t,D)\Lambda^{-1}\langle\Lambda\rangle p_0\big\|_{L_t^\infty L^2}+\big\|\big(m_1(t,D)+2m_2(t,D)\big)q_0\big\|_{L_t^\infty H^1}\\&\quad\lesssim\|p_0\|_{L^2}+\|q_0\|_{H^1}.\end{aligned}\tag{3.22}$$

Now we deal with the third term in (3.14). Applying (3.11)–(3.13) and (2.7) with $(r,\rho,s)=(2,\infty,1)$ and $m(t,\xi)=m_1(t,\xi)$ to G, we get

$$\left\|\int_0^t m_1(t-\tau,D)\Lambda G(\tau)d\tau\right\|_{L_t^\infty L^2}=\left\|\int_0^t m_1(t-\tau,D)G(\tau)d\tau\right\|_{L_t^\infty\dot H^1}\lesssim\|G\|_{L_t^2L^2}.\tag{3.23}$$

It remains to control $\mathfrak{F}_2(p,q)$. By partition of unit, i.e., $G=G^l+G^m+G^h$, and Lemma 2.1, we get

$$\begin{aligned}
&\left\| \int_0^t m_2(t-\tau, D)G(\tau)d\tau \right\|_{L_t^\infty H^1} \\
&= \left\| \int_0^t m_2(t-\tau, D)\langle \Lambda \rangle G(\tau)d\tau \right\|_{L_t^\infty L^2} \\
&\le \left\| \int_0^t \left(m_2(t-\tau, D)\Lambda^{-1}\right)\Lambda\langle \Lambda \rangle \left(G^l + G^m\right)d\tau + m_2(t-\tau, D)\langle \Lambda \rangle G^h d\tau \right\|_{L_t^\infty L^2} \\
&\le \left\| \int_0^t m_2(t-\tau, D)\Lambda^{-1}\langle \Lambda \rangle \left(G^l + G^m\right)d\tau \right\|_{L_t^\infty \dot{H}^1} + \left\| \int_0^t m_2(t-\tau, D)G^h d\tau \right\|_{L_t^\infty H^1} \\
&:= I_{11} + I_{12}.
\end{aligned}$$

Applying (3.11)–(3.13), (2.7) and Bernstein inequalities in Lemma 2.1 to I_{11} and I_{12}, we obtain that

$$\begin{aligned}
I_{11} &= \left\| \int_0^t m_2(t-\tau, D)\Lambda^{-1}\langle \Lambda \rangle \left(G^l + G^m\right)d\tau \right\|_{L_t^\infty \dot{H}^1} \\
&\le \left\| \langle \Lambda \rangle \left(G^l + G^m\right) \right\|_{L_t^2 L^2} \lesssim \left\| \langle \cdot \rangle \eta(\cdot) + \langle \cdot \rangle \varphi(\cdot) \right\|_{L_\xi^\infty} \|G\|_{L_t^2 L^2} \lesssim \|G\|_{L_t^2 L^2},
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
I_{12} &= \left\| \int_0^t m_2(t-\tau, D)\langle \Lambda \rangle G^h d\tau \right\|_{L_t^\infty L^2} \\
&\le \left\| \Lambda^{-1}\langle \Lambda \rangle \psi(D) G \right\|_{L_t^1 \dot{H}^1} \le \left\| \left\| |\cdot|^{-1} \langle \cdot \rangle \psi(\cdot) \right\|_{L_\xi^\infty} \|G\|_{\dot{H}^1} \right\|_{L_t^1} \lesssim \|G\|_{L_t^1 \dot{H}^1}
\end{aligned} \tag{3.25}$$

where in (3.25) we used $0 \le \frac{\langle \xi \rangle \psi(\xi)}{|\xi|} \le 2$.

Part II. *Estimate of* $\|\mathfrak{F}(p,q)\|_{L_t^2 \dot{H}^1 \times L_t^2 \dot{H}^1}$. In order to estimate $\mathfrak{F}_1(p,q)$, we get from (3.11)–(3.13) that $|\xi| m_1(t,\xi) + 2m_2(t,\xi) \in L_t^2 L_\xi^\infty$. Then by applying (2.5) with $(r,s) = (2,1)$ to $m_1(t,D)p_0 + 2m_2(t,D)\Lambda^{-1}q_0$, we get

$$\left\| m_1(t,D)p_0 + 2m_2(t,D)\Lambda^{-1}q_0 \right\|_{L_t^2 \dot{H}^1} \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2}. \tag{3.26}$$

From (3.11)–(3.13) and (3.21), we have $|\xi| m_1(t,\xi) \in L_t^\infty L_\xi^2$ and $m_2(t,\xi) \in L_\xi^\infty L_t^2$. Then applying (2.5) with $(r,s) = (2,1)$ to $m_2(t,D)\Lambda^{-1}p_0 + (m_1(t,D) + 2m_2(t,D))q_0$ gives

$$\begin{aligned}
&\left\| -2m_2(t,D)\Lambda^{-1}p_0 + m_1(t,D)q_0 + 2m_2(t,D)q_0 \right\|_{L_t^2 \dot{H}^1} \\
&\quad \lesssim \|p_0\|_{L^2} + \|q_0\|_{L^2} + \|\Lambda q_0\|_{L^2} \lesssim \left\| (p_0, q_0) \right\|_{L^2 \times H^1}.
\end{aligned} \tag{3.27}$$

Next we will deal with the third term on the r.h.s. of (3.14). Applying (3.11)–(3.13) and (2.7) with $r = \rho = s = 2$ and $m(t,\xi) = m_1(t,\xi)$ to G, we get

$$\left\| \int_0^t m_1(t-\tau, D)\Lambda G(\tau)d\tau \right\|_{L_t^2 \dot{H}^1} = \left\| \int_0^t m_1(t-\tau, D)G(\tau)d\tau \right\|_{L_t^2 \dot{H}^2} \lesssim \|G\|_{L_t^2 L^2}. \tag{3.28}$$

Thus it remains to control $\mathfrak{F}_2(p, q)$. Following the similar arguments of (3.24)–(3.25), we get

$$\left\| \int_0^t m_2(t-\tau, D)G(\tau)d\tau \right\|_{L_t^2 \dot{H}^1} = \left\| \int_0^t m_2(t-\tau, D)\Lambda G(\tau)d\tau \right\|_{L_t^2 L^2}$$
$$\lesssim \|G\|_{L_t^2 L^2} + \left\|G^h\right\|_{L_t^1 \dot{H}^1} \lesssim \|G\|_{L_t^2 L^2} + \|G\|_{L_t^1 \dot{H}^1}. \tag{3.29}$$

Part III. *Estimate of* $\|\mathfrak{F}_1(p, q)\|_{L_t^1 \dot{H}_\psi^{7/4}}$. From maximal regularity results, (3.14) and (3.16), we observe that

$$\begin{aligned}
\left\|\mathfrak{F}_1(p, q)\right\|_{L_t^1 \dot{H}_\psi^{\frac{7}{4}}} &\le \left\|m_1(t, D)p_0\right\|_{L_t^1 \dot{H}_\psi^{\frac{7}{4}}} + 2\left\|m_2(t, D)\Lambda^{-1}q_0\right\|_{L_t^1 \dot{H}_\psi^{\frac{7}{4}}} \\
&\quad + \left\| \int_0^t m_1(t-\tau)\Lambda G(\tau)d\tau \right\|_{L_t^1 \dot{H}_\psi^{\frac{7}{4}}} \\
&:= I_{21} + I_{22} + I_{23}.
\end{aligned} \tag{3.30}$$

As for I_{21}, by applying (3.11)–(3.12) and Lemma 2.1 to I_{21} with $|\xi| \ge 2^4$, we claim that

$$\begin{aligned}
I_{21} &= \left\|m_1(t, D)p_0\right\|_{L_t^1 \dot{H}_\psi^{\frac{7}{4}}} \\
&\lesssim \left\|e^{-ct|\xi|^2}|\xi|^{\frac{7}{4}}\psi(\xi)\widehat{p}_0\right\|_{L_t^1 L_\xi^2} + \left\|e^{-ct}\psi(\xi)|\xi|^{\frac{7}{4}}\langle\xi\rangle^{-2}\right\|_{L_t^1 L_\xi^\infty}\|\widehat{p}_0\|_{L_\xi^2} \\
&\lesssim \|p_0\|_{L^2}.
\end{aligned} \tag{3.31}$$

In order to show (3.31), it suffices to estimate $\|e^{-ct|\xi|^2}|\xi|^{\frac{7}{4}}\psi(\xi)\widehat{p}_0\|_{L_t^1 L_\xi^2}$. In fact,

$$\begin{aligned}
\left\|e^{-ct|\xi|^2}|\xi|^{\frac{7}{4}}\psi(\xi)\widehat{p}_0\right\|_{L_t^1 L_\xi^2} &= \int_0^1 \left(\int_{\mathbb{R}^3} e^{-2ct|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi)|\widehat{p}_0|^2 d\xi \right)^{\frac{1}{2}} dt \\
&\quad + \int_1^\infty \left(\int_{|\xi|>2^4} e^{-2ct|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi)|\widehat{p}_0|^2 d\xi \right)^{\frac{1}{2}} dt \\
&:= I_{211} + I_{212}.
\end{aligned}$$

Applying the Plancherel equality to I_{211}, we have $\sup_{\xi\in\mathbb{R}^3} e^{-2ct|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi) \lesssim t^{-\frac{7}{4}}$ and

$$I_{211} = \int_0^1 \left(\int_{\mathbb{R}^3} e^{-2ct|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi)|\widehat{p_0}|^2 d\xi \right)^{\frac{1}{2}} dt \lesssim \int_0^1 t^{-\frac{7}{8}} dt \|\widehat{p_0}\|_{L^2_\xi} \lesssim \|p_0\|_{L^2}.$$

If $|\xi| > 2^4$ and $t > 1$, we get $e^{-2ct|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi) \le e^{-ct}e^{-c|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi) \lesssim e^{-ct}$ and

$$I_{212} \lesssim \int_1^\infty \left(\int_{|\xi|>2^4} e^{-ct}e^{-c|\xi|^2}|\xi|^{\frac{7}{2}}\psi(\xi)|\widehat{p_0}|^2 d\xi \right)^{\frac{1}{2}} dt \lesssim \int_1^\infty e^{-ct} dt \|\widehat{p_0}\|_{L^2_\xi} \lesssim \|p_0\|_{L^2}.$$

Similarly, by applying (3.11)–(3.12) and Lemma 2.1 to I_{22}, we get

$$I_{22} = \left\| m_2(t,D)\Lambda^{-1}q_0 \right\|_{L^1_t \dot{H}^{\frac{7}{4}}_\psi} \lesssim \left\| e^{-ct}\psi(\xi) \right\|_{L^1_t L^\infty_\xi} \|q_0\|_{\dot{H}^{\frac{3}{4}}} \lesssim \|q_0\|_{H^1}. \tag{3.32}$$

It remains to estimate I_{23}. Since $m_1(t-\tau,\xi)|\xi|^{\frac{7}{4}} \lesssim (t-\tau)^{-\frac{7}{8}}$, $m_1(t-\tau,\xi)|\xi|^{\frac{11}{4}} \lesssim (t-\tau)^{-\frac{11}{8}}$, we get

$$\begin{aligned}
I_{23} &= \left\| \int_0^t m_1(t-\tau)\Lambda G(\tau)d\tau \right\|_{L^1_t \dot{H}^{\frac{7}{4}}_\psi} = \left\| \int_0^t m_1(t-\tau)\Lambda^{\frac{7}{4}}\Lambda\psi(D)G(\tau)d\tau \right\|_{L^1_t L^2} \\
&\lesssim \int_0^\infty \int_0^t \min\left\{(t-\tau)^{-\frac{7}{8}} \left\|\Lambda\psi(D)G(\tau)\right\|_{L^2},\ (t-\tau)^{-\frac{11}{4}} \left\|\psi(D)G\right\|_{L^2}\right\} d\tau dt \\
&\lesssim \int_0^\infty \int_0^t \min\left\{(t-\tau)^{-\frac{7}{8}},\ (t-\tau)^{-\frac{11}{4}}\right\} \left\|G(\tau)\right\|_{\dot{H}^1} d\tau dt \\
&\lesssim \int_0^\infty \int_\tau^\infty \min\left\{(t-\tau)^{-\frac{7}{8}}, (t-\tau)^{-\frac{11}{4}}\right\} dt \left\|G(\tau)\right\|_{\dot{H}^1} d\tau \\
&\lesssim \int_0^\infty \int_0^\infty \min\left\{t^{-\frac{7}{8}},\ t^{-\frac{11}{4}}\right\} dt \left\|G(\tau)\right\|_{\dot{H}^1} d\tau \\
&\lesssim \|G\|_{L^1_t \dot{H}^1}
\end{aligned} \tag{3.33}$$

where in the fourth inequality we have applied (2.3) to $\psi(D)G$ with $(s,a) = (1,2)$.

Combining the above arguments, we finish the proof. □

Recalling $G = \Lambda^{-1}\nabla\cdot(p\nabla\Lambda^{-1}q)$ and boundedness of the Riesz transforms in L^2, we only need to control $\|\nabla\cdot(p\nabla\Lambda^{-1}q)\|_{L^1_t L^2} = \|G\|_{L^1_t \dot{H}^1}$.

The following key lemma is devoted to estimating $\|\nabla p\cdot\nabla\Lambda^{-1}q\|_{L_t^1L^2}$ and $\|p\Lambda q\|_{L_t^1L^2}$, where

$$\nabla\cdot\big(p\nabla\Lambda^{-1}q\big)=\nabla p\cdot\nabla\Lambda^{-1}q-p\Lambda q. \tag{3.34}$$

Lemma 3.2. *Let $X\times Y$ be defined as in (3.17). If $u\in X$ and $v\in Y$, then we get*

$$\|u\nabla v\|_{L_t^1L^2}+\|\nabla uv\|_{L_t^1L^2}\lesssim\|u\|_{L_t^2\dot H^1}\|v\|_{L_t^2\dot H^1}+\|u\|_{L_t^1\dot H_\psi^{7/4}}\|v\|_{L_t^\infty H^1}, \tag{3.35}$$

$$\|uv\|_{L_t^2L^2}\lesssim\|u\|_{L_t^2\dot H^1}\|v\|_{L_t^\infty H^1}. \tag{3.36}$$

Proof. At first, we prove (3.35). Since $u\nabla v=(u^l+u^m)\nabla v+u^h\nabla v$, by Hölder's inequality, we have

$$\begin{aligned}\|u\nabla v\|_{L_t^1L^2}&\le\big\|(u^l+u^m)\nabla v\big\|_{L_t^1L^2}+\big\|u^h\nabla v\big\|_{L_t^1L^2}\\&\lesssim\big\|u^l+u^m\big\|_{L_t^2L^\infty}\|\nabla v\|_{L_t^2L^2}+\big\|u^h\big\|_{L_t^1L^\infty}\|\nabla v\|_{L_t^\infty L^2}\\&:=I_{31}+I_{32}.\end{aligned}$$

From (2.1) with $|\xi|<2^5$ and the Sobolev embedding theorem, it is easy to prove that

$$\begin{aligned}I_{31}&=\big\|u^l+u^m\big\|_{L_t^2L^\infty}\|\nabla v\|_{L_t^2L^2}\lesssim\big\|u^l+u^m\big\|_{L_t^2L^6}\|v\|_{L_t^2\dot H^1}\\&\lesssim\big\|u^l+u^m\big\|_{L_t^2\dot H^1}\|v\|_{L_t^2\dot H^1}\lesssim\|u\|_{L_t^2\dot H^1}\|v\|_{L_t^2\dot H^1}\end{aligned} \tag{3.37}$$

where in the fourth inequality, we used the fact that $\eta(\xi)+\varphi(\xi)$ is an L^2-multiplier. From (2.3) with $|\xi|>2^4$ and the Sobolev embedding theorem $H^{\frac{7}{4}}\hookrightarrow L^\infty$, we get

$$I_{32}=\big\|u^h\big\|_{L_t^1L^\infty}\|\nabla v\|_{L_t^\infty L^2}\lesssim\big\|u^h\big\|_{L_t^1H^{7/4}}\|v\|_{L_t^\infty H^1}\lesssim\|u\|_{L_t^1\dot H_\psi^{7/4}}\|v\|_{L_t^\infty H^1}. \tag{3.38}$$

Estimate of $\|\nabla uv\|_{L_t^1L^2}$ is rather simple. Indeed, by making use of Hölder's inequality, we get

$$\|\nabla uv\|_{L_t^1L^2}\lesssim\|\nabla u\|_{L_t^2L^3}\|v\|_{L_t^2L^6}\lesssim\|u\|_{L_t^2\dot H^1}\|v\|_{L_t^2\dot H^1}. \tag{3.39}$$

Hence we prove (3.35).

It remains to prove (3.36). By making use of Hölder's inequality, we get

$$\|uv\|_{L_t^2L^2}\lesssim\|u\|_{L_t^2L^6}\|v\|_{L_t^\infty L^3}\lesssim\|u\|_{L_t^2\dot H^1}\|v\|_{L_t^\infty H^1}. \tag{3.40}$$

Finally, combining (3.37)–(3.40), we prove all the desired results. □

Applying (3.35) and (3.36) to $\nabla p\cdot\nabla\Lambda^{-1}q-p\Lambda q$ and $\Lambda^{-1}\nabla\cdot(p\nabla\Lambda^{-1}q)$, respectively, combining Proposition 3.1, Lemma 3.2 and (3.17), we have the following *a priori* estimates.

Corollary 3.3. *Let* (p,q) *be a solution to system* (1.7) *with initial data* $(p_0,q_0)\in L^2\times H^1$ *and* $\mathfrak{F}$ *be defined as in* (3.16). *Then there holds*

$$\big\|\mathfrak{F}(p,q)\big\|_{X\times Y}\lesssim\big\|(p_0,q_0)\big\|_{L^2\times H^1}+\big\|(p,q)\big\|^2_{X\times Y}.$$

Proof of Theorem 1.1. Applying Lemma 2.4, Corollary 3.3 and following a standard fixed point argument, we prove Theorem 1.1 when $\|(p_0,q_0)\|_{L^2\times H^1}$ is small. □

3.3. *Proof of Theorem 1.2*

In this subsection, we first prove the *a priori* estimates including the crucial bilinear estimates.

Proposition 3.4. *Let* (p,q) *be a solution to system* (1.7) *with initial data* $(p_0,q_0)\in H^2\times H^1$ *and* $\mathfrak{F}$ *be defined as in* (3.16). *Then there holds*

$$\big\|\mathfrak{F}(p,q)\big\|_{L_t^\infty H^2\times L_t^\infty H^1}\lesssim\big\|(p_0,q_0)\big\|_{H^2\times H^1}+\|p\|_{L_t^\infty H^2}\|q\|_{L_t^\infty H^1}. \tag{3.41}$$

Proof. We first derive the estimate for $\mathfrak{F}_1(p,q)$ as defined in (3.14)–(3.16). By applying $m_1(t,\xi)$, $\frac{m_2(t,\xi)}{|\xi|}\in L_t^\infty L_\xi^\infty$, $\frac{\langle\xi\rangle\psi(\xi)}{|\xi|}\in L_\xi^\infty$ and (2.4) for $(r,s)=(\infty,0)$ to $m_1(t,D)p_0+2m_2(t,D)\Lambda^{-1}q_0$, we get

$$\begin{aligned}\big\|m_1(t,D)p_0+2m_2(t,D)\Lambda^{-1}q_0\big\|_{H^2}&\lesssim\big\|m_1(t,\xi)\langle\xi\rangle^2\widehat{p_0}\big\|_{L_\xi^2}+\bigg\|\frac{m_2(t,\xi)\langle\xi\rangle^2}{|\xi|}\widehat{q_0}\bigg\|_{L_\xi^2}\\&\lesssim\|p_0\|_{H^2}+\|q_0\|_{H^1}.\end{aligned} \tag{3.42}$$

Similarly, using $m_1(t,\xi)+2m_2(t,\xi)$, $\frac{\langle\xi\rangle m_2(t,\xi)}{|\xi|}\in L_t^\infty L_\xi^\infty$ and applying (2.4) with $(r,s)=(\infty,1)$ to $m_2(t,D)\Lambda^{-1}p_0+(m_1(t,D)+2m_2(t,D))q_0$, we get

$$\big\|2m_2(t,D)\Lambda^{-1}p_0-\big(m_1(t,D)+2m_2(t,D)\big)q_0\big\|_{L_t^\infty H^1}\lesssim\|p_0\|_{L^2}+\|q_0\|_{H^1}. \tag{3.43}$$

Next we deal with the third term on the r.h.s. of (3.14). Applying (3.11)–(3.13) and (2.7) with $(r,\rho,s)=(2,\infty,1)$ and $m(t,\xi)=m_1(t,\xi)$ to G, we get

$$\Bigg\|\int_0^t m_1(t-\tau,D)G(\tau)d\tau\Bigg\|_{L_t^\infty\dot H^1}\lesssim\|G\|_{L_t^\infty\dot H^{-1}}\lesssim\|G\|_{L_t^\infty L^{\frac{3}{2}}}\lesssim\|p\|_{L_t^\infty H^2}\|q\|_{L_t^\infty H^1} \tag{3.44}$$

and

$$\begin{aligned}\Bigg\|\int_0^t m_1(t-\tau,D)\Lambda G(\tau)d\tau\Bigg\|_{L_t^\infty\dot H^2}&=\Bigg\|\int_0^t m_1(t-\tau,D)G(\tau)d\tau\Bigg\|_{L_t^\infty\dot H^3}\lesssim\|G\|_{L_t^\infty\dot H^1}\\&\lesssim\|\nabla p\|_{L_t^\infty L^6}\|q\|_{L_t^\infty L^3}+\|p\|_{L_t^\infty L^\infty}\|\nabla q\|_{L_t^\infty L^2}\\&\lesssim\|p\|_{L_t^\infty H^2}\|q\|_{L_t^\infty H^1}.\end{aligned} \tag{3.45}$$

It remains to control $\mathfrak{F}_2(p,q)$. By following the similar argument as in (3.24) and (3.25), we get

$$\left\| \int_0^t m_2(t-\tau,D)\Lambda^{-1}\langle\Lambda\rangle\left(G^l+G^m\right)d\tau \right\|_{L_t^\infty \dot{H}^1} \lesssim \|G\|_{L_t^\infty \dot{H}^{-1}} \lesssim \|G\|_{L_t^\infty L^{\frac{3}{2}}}, \tag{3.46}$$

$$\left\| \int_0^t m_2(t-\tau,D)\langle\Lambda\rangle G^h d\tau \right\|_{L_t^\infty L^2} \lesssim \|G\|_{L_t^\infty H^1}, \tag{3.47}$$

where we have used the damping property of G^h, i.e., $\psi(\xi)m_2(t,\xi) \lesssim e^{-ct}$.

Combining the above arguments, we finish the proof. □

The following proposition is used to prove decay estimates of solutions to (1.1).

Proposition 3.5. *Let (p,q) be a solution to system (1.7) with initial data $(p_0,q_0) \in H^2 \times H^1$ and $\mathfrak{F}$ be defined as in (3.16). Then for any $t>0$, there holds*

$$\begin{aligned}
&(1+t)^{\frac{1}{2}}\left\|\nabla\mathfrak{F}(p,q)\right\|_{L^2} + (1+t)^{\frac{7}{8}}\left\|\Lambda^{\frac{7}{4}}\mathfrak{F}_1(p,q)\right\|_{L^2} \\
&\quad \lesssim \left\|(p_0,q_0)\right\|_{H^2\times H^1} + \sup_{t>0}\left((1+t)^{\frac{1}{2}}\left\|(\nabla p,\nabla q)\right\|_{L^2\times L^2}\right)^2 + \sup_{t>0}\left((1+t)^{\frac{7}{8}}\left\|\Lambda^{\frac{7}{4}}p\right\|_{L^2}\right)^2.
\end{aligned}$$

Proof. Noticing that $m_1(t,\xi)|\xi| + m_2(t,\xi) \lesssim e^{-ct|\xi|^2}|\xi| + e^{-ct}$, then we have

$$\begin{aligned}
\left\|m_1(t,D)\Lambda p_0 + 2m_2(t,D)q_0\right\|_{L^2} &\lesssim \left\|m_1(t,\xi)|\xi|\widehat{p_0}\right\|_{L^2_\xi} + \left\|m_2(t,\xi)\widehat{q_0}\right\|_{L^2_\xi} \\
&\lesssim (1+t)^{-\frac{1}{2}}\left(\|p_0\|_{H^1} + \|q_0\|_{H^1}\right)
\end{aligned}$$

and $\|m_1(t,D)\Lambda^{\frac{7}{4}}p_0 + 2m_2(t,D)\Lambda^{\frac{3}{4}}q_0\|_{L^2} \lesssim (1+t)^{-\frac{7}{8}}(\|p_0\|_{H^2} + \|q_0\|_{H^1})$ for all $t>0$.

Similarly, for all $t>0$, we have

$$\left\|2m_2(t,D)p_0 - \left(m_1(t,D)\Lambda + 2m_2(t,D)\Lambda\right)q_0\right\|_{L^2} \lesssim (1+t)^{-\frac{1}{2}}\left(\|p_0\|_{H^1} + \|q_0\|_{H^1}\right).$$

To control the third term on the r.h.s. of (3.14), by using $m_1(t,\xi)|\xi| \lesssim e^{-ct|\xi|^2}|\xi| + e^{-ct}$, chain rule, Plancherel identity and Sobolev embedding, for any $t>0$, we get

$$\begin{aligned}
&\left\| \int_0^t m_1(t-\tau,D)\Delta G(\tau)d\tau \right\|_{L^2} \\
&\quad \lesssim \int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}}d\tau \sup_{\tau>0}(1+\tau)^{\frac{5}{4}}\left(\|\nabla p\|_{L^3}\|q\|_{L^6} + \|\nabla q\|_{L^2}\|p\|_{L^\infty}\right) \\
&\quad \lesssim (1+t)^{-\frac{1}{2}}\sup_{\tau>0}\left((1+\tau)^{\frac{3}{4}}\left\|\Lambda^{\frac{3}{2}}p\right\|_{L^2} + (1+\tau)^{\frac{7}{4}}\left\|\Lambda^{\frac{7}{4}}p\right\|_{L^2}\right)(1+\tau)^{\frac{1}{2}}\|\nabla q\|_{L^2}.
\end{aligned}$$

Similarly, for all $t > 0$, we have

$$\left\| \int_0^t m_1(t-\tau, D)\Lambda^{\frac{7}{4}} \nabla G(\tau) d\tau \right\|_{L^2}$$

$$\lesssim \int_0^t (t-\tau)^{-\frac{7}{8}}(1+\tau)^{-\frac{5}{4}} d\tau \sup_{\tau>0}(1+\tau)^{\frac{5}{4}}\left(\|\nabla p\|_{L^3}\|q\|_{L^6} + \|\nabla q\|_{L^2}\|p\|_{L^\infty}\right)$$

$$\lesssim (1+t)^{-\frac{7}{8}} \sup_{\tau>0}\left((1+\tau)^{\frac{3}{4}}\left\|\Lambda^{\frac{3}{2}} p\right\|_{L^2} + (1+\tau)^{\frac{7}{4}}\left\|\Lambda^{\frac{7}{4}} p\right\|_{L^2}\right)(1+\tau)^{\frac{1}{2}}\|\nabla q\|_{L^2},$$

$$\left\| \int_0^t m_2(t-\tau, D)\nabla G d\tau \right\|_{L^2}$$

$$\lesssim \int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{4}} d\tau \sup_{\tau>0}(1+\tau)^{\frac{5}{4}}\left(\|\nabla p\|_{L^3}\|q\|_{L^6} + \|\nabla q\|_{L^2}\|p\|_{L^\infty}\right)$$

$$\lesssim (1+t)^{-\frac{1}{2}} \sup_{\tau>0}\left((1+\tau)^{\frac{3}{4}}\left\|\Lambda^{\frac{3}{2}} p\right\|_{L^2} + (1+\tau)^{\frac{7}{4}}\left\|\Lambda^{\frac{7}{4}} p\right\|_{L^2}\right)(1+\tau)^{\frac{1}{2}}\|\nabla q\|_{L^2}.$$

Combining the above arguments and $\|\Lambda^{\frac{3}{2}} p\|_{L^2} \lesssim \|\Lambda^{\frac{7}{4}} p\|_{L^2}^{\frac{2}{3}} \|\nabla p\|_{L^2}^{\frac{1}{3}}$, we finish the proof. □

Proof of Theorem 1.2. Applying Lemma 2.4, Propositions 3.4 and 3.5, following standard fixed point argument, we prove Theorem 1.2 when $\|(p_0, q_0)\|_{L^2 \times H^1}$ is small. □

3.4. Proof of Corollary 1.3

Proof of Corollary 1.3. Applying Lemma 2.4 and Corollary 3.3 to system (1.1), we prove the existence results of Corollary 1.3. As for the decay property of v, one can use (1.20)–(1.21). Hence we omit the details. □

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