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Largest well-posed spaces for the general diffusion system with nonlocal interactions

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A B S T R A C T

The authors derive a general diffusion (GD) system with nonlocal interactions of special structure via energetic variational approach and observe that there exist two critical values of s, i.e. $s = \frac{1}{2}, 1$, for the nonlocal interactions, where $s = \frac{1}{2}$ reflects how strong nonlocal property we have and $s = 1$ affects the linearization and choice of initial data spaces. The authors also establish the global existence and uniqueness of mild solution.

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1. Introduction

We study the following N-dimensional ($N \geq 2$) general diffusion system

$$\begin{cases} \rho_t + \nabla \cdot (u\rho) & = 0, \\ \mu_{en} \nabla \rho + \mu_{in}\, \rho \nabla K_{2s} * \rho & = -u\rho, \end{cases} \tag{1.1}$$

where[1] $0 \leq s \leq 1$, $0 < \mu_{en} \leq \infty$, $-\infty < \mu_{in} < \infty$, u is the effective transport velocity vector, and $u\rho$ is the flux that contains nonlocal term $\nabla K_{2s} * \rho$ with $\nabla K_{2s} * \rho = \mathcal{F}^{-1}(i\xi|\xi|^{-2s}\mathcal{F}\rho(\xi))$ in distributional sense, see [11, Chapter 2].

The model arises from the consideration of a continuum density distribution ρ that evolves in time following a velocity field u, according to the continuity equation $\rho_t + \nabla \cdot (u\rho) = 0$ with

$$\int \rho(x,t)dx = \int \rho(x,t)|_{t=0}\, dx$$

for all $t > 0$. Here u is given by the following potential

$$u = -\mu_{en} \nabla \ln \rho - \mu_{in} \nabla K_{2s} * \rho,$$

which arises, for instance, in porous media for $\mu_{in} > 0$ and $s = 0$ according to Darcy's law [6] and chemotaxis for $\mu_{in} < 0$ and $s = 1$ [8,16], respectively.

1.1. Energetic variational approach

In this subsection, we employ the Energetic Variational Approach (EVA) [13] for an isothermal closed system. Hence we can derive from the First Law and Second Law of Thermodynamics the following energy dissipation law:

$$\frac{d}{dt} E^{total} = -\Delta, \tag{1.2}$$

where E^{total} represents the sum of kinetic energy and total Helmholtz free energy, and Δ is the energy dissipation rate/entropy production. As a direct consequence of the choice of total energy functional, dissipation functional, and kinematic relation of the variables employed in the system, one can get all the physics and the assumptions correspondingly.

As a precise framework, one can use the EVA to obtain the force balance equations from the general dissipation law (1.2). Precisely speaking, the Least Action Principle (LAP) determines the Hamiltonian part, and the Maximum Dissipation Principle (MDP) gives the dissipative part. Formally, LAP states that work equals force multiplies distance, i.e.

[1] One of the special cases of $s > 1$, i.e. $s = 2$ and $N = 4$ is also considered.

$$\delta E = \text{force} \times \delta x,$$

where δ is the variation in general sense and x is the position. This gives the Hamiltonian part of the system and the conservative force [1,2], while MDP, by Onsager [28,29], giving the dissipative force

$$\delta \frac{1}{2}\Delta = \text{force} \times \delta x_t,$$

where the factor "$\frac{1}{2}$" is consistent with the choice of quadratic form of the dissipation rate of energy, which in turn describes the linear response theory for long-time near equilibrium dynamics [19]. For instance, we first consider system (1.1) with $s = 0$, i.e.

$$\rho_t = \nabla \cdot (\mu_{en} \nabla \rho + \mu_{in}\, \rho \nabla \rho). \tag{1.3}$$

Let us start with the energy dissipation law with prescribed Helmholtz/free energy and entropy production functionals

$$\frac{d}{dt}\int \Big(\mu_{en}\, \rho \ln \rho + \frac{1}{2}\mu_{in}\, \rho^2\Big) dx = -\int \rho |u|^2 dx, \tag{1.4}$$

where $u(x(X,t),t) = x_t(X,t)$, $x(X,t)$ is the flow map, X is the reference coordinate, and the kinematic relation is just the conservation of mass

$$\rho_t + \nabla \cdot (\rho\, u) = 0. \tag{1.5}$$

Let $\mathcal{A} = \int_\Omega (w_1(\rho) + w_2(\rho))dx = \int_\Omega \mu_{en}\, \rho \ln \rho + \frac{1}{2}\mu_{in}\, \rho^2 dx$ and $\Delta = \int \rho |u|^2 dx$. By using the force balance law between conservative and dissipative forces, we get

$$\frac{\delta \mathcal{A}}{\delta x} = -\rho u = -\frac{1}{2}\frac{\delta \Delta}{\delta u}.$$

In fact, $\frac{\delta \mathcal{A}}{\delta x} = \nabla\Big(\mu_{en}\rho + \frac{1}{2}\mu_{in}\rho^2\Big)$ since

$$\delta A = \int_\Omega \sum_{j=1,2} \nabla\Big(w_j'(\rho)\rho - w_j(\rho)\Big) \cdot \delta x\, dx = \int_\Omega \nabla\Big(\mu_{en}\rho + \frac{1}{2}\mu_{in}\rho^2\Big) \cdot \delta x\, dx.$$

Therefore, $\nabla(\mu_{en}\rho + \frac{1}{2}\mu_{in}\rho^2) = -\rho\, u$, which together with (1.5) yields (1.3).

For any $s \in (0,1]$, following the similar argument of energetic variational approach, we also start with the energy law

$$\frac{d}{dt}\Big[\int \mu_{en}\rho \ln \rho dx + \int \frac{1}{2}\mu_{in}\rho(K_{2s} * \rho) dx\Big] = -\int \rho |u|^2 dx. \tag{1.6}$$

According to EVA, the total energy E^{total} and the dissipation Δ are

$$E^{total} = \int \mu_{en}\rho \ln \rho \, dx + \int \frac{1}{2}\mu_{in}\, \rho(K_{2s} * \rho) dx, \quad \Delta = \int \rho |u|^2 dx.$$

Define the action functional $\mathcal{A}$ of entropy and internal energy as

$$\mathcal{A} = \int_\Omega \mu_{en}\rho \ln \rho \, dx + \int_\Omega \frac{1}{2}\mu_{in}\rho K_{2s} * \rho \, dx. \tag{1.7}$$

By making use of flow map $x(X,t)$, taking variation of $\mathcal{A}$ with respect to x, taking variation of Δ with respect u, and using the force balance law, we get

$$\frac{\delta \mathcal{A}}{\delta x} = \mu_{en}\nabla\rho + \mu_{in}\rho\nabla K_{2s} * \rho = -\rho u = -\frac{1}{2}\frac{\delta \Delta}{\delta u}. \tag{1.8}$$

Finally, plugging identity (1.8) into equation (1.5) gives system (1.1).

1.2. *Linearization near positive constant state*

In this subsection, we aim at showing the *difference* between $\mu_{in} > 0$ and $\mu_{in} < 0$. The key idea is linearization of ρ of system (1.1) near some positive constant $\bar{\rho}_0$ such that ρ has positive lower and upper bounds, which guarantees the *nonpositivity* of the "right hand side of (1.6)".

Assume that $0 \le s < \frac{1}{2}$. By using Fourier/inverse Fourier transformation and $\xi|\xi|^{-2s} \times$ "delta function" $= 0$, one gets

$$\nabla K_{2s} * (\tilde{\rho} + \bar{\rho}_0) = \mathcal{F}^{-1}\Big(i\xi|\xi|^{-2s}\mathcal{F}\tilde{\rho}(\xi)\Big) = \nabla K_{2s} * \tilde{\rho} \tag{1.9}$$

for any tempered distribution ρ. Linearizing system (1.1) near any constant state $\bar{\rho}_0$ (> 0) yields[2]

$$\tilde{\rho}_t = \Big(\mu_{en}\Delta - \mu_{in}\bar{\rho}_0(-\Delta)^{1-s}\Big)\tilde{\rho} + \mu_{in}\nabla \cdot \Big(\tilde{\rho}\nabla K_{2s} * \tilde{\rho}\Big), \tag{1.10}$$

where $\mathcal{F}\big(\mu_{en}\Delta - \mu_{in}\bar{\rho}_0(-\Delta)^{1-s}\big) = -\mu_{en}|\xi|^2 - \mu_{in}\bar{\rho}_0|\xi|^{2-2s}$.

From (1.10) it is clear that: if $\mu_{in} > 0$ and $\mu_{en} = 0$, then $-\mu_{in}\bar{\rho}_0|\xi|^{2-2s}$ gives fractional dissipation; else if $\mu_{in}, \mu_{en}, s > 0$, then $-\mu_{en}|\xi|^2 - \mu_{in}\bar{\rho}_0|\xi|^{2-2s}$ gives different dissipations for high/low frequency; else if $\mu_{in} < 0$ and $\mu_{en} > 0$, then $-\mu_{en}|\xi|^2 - \mu_{in}\bar{\rho}_0|\xi|^{2-2s}$ becomes positive for sufficiently small frequency which gives us no dissipation and might produce finite time blow-up solution.

[2] Small solution to (1.10) with large positive $\bar{\rho}_0$ can generate large positive solution to (1.1) with infinite mass.

Assume that $\frac{1}{2} \le s < 1$. In this case, it seems difficult to get (1.10) near $\bar{\rho}_0\,(>0)$ since $\nabla K_{2s} * \bar{\rho}_0$ is not well-defined even in distributional sense. However, $\nabla\cdot(\nabla K_{2s}*(\tilde{\rho}+\bar{\rho}_0)) = \Delta K_{2s} * \tilde{\rho}$ for any tempered distribution $\tilde{\rho}$. Then system (1.1) becomes

$$\tilde{\rho}_t = \Big(\mu_{en}\Delta - \mu_{in}\bar{\rho}_0(-\Delta)^{1-s}\Big)\tilde{\rho} + \mu_{in}\nabla\cdot\Big(\tilde{\rho}\nabla K_{2s} * (\tilde{\rho}+\bar{\rho}_0)\Big). \tag{1.11}$$

Therefore, from [31, Remark 3, p. 239], we have study system (1.11) for distribution modulo polynomials, which shows that homogeneous Besov space is a natural choice. Similar arguments are applied for μ_{en} and μ_{in}.

Assume that $1 \le s \le \frac{N}{2}$. In this case, we also need to study system (1.1) in distribution modulo polynomial sense. Therefore, it suffices to study small data Cauchy problem.

Conclusively, for $0 \le s < 1$ we can observe the difference of $\mu_{in} > 0$ and $\mu_{in} < 0$ by doing linearization; for $1 \le s \le \frac{N}{2}$ we are unable to show their difference since we work for small data problem in homogeneous Besov spaces (subset of tempered distribution modulo polynomials). *Therefore, $s = 1$ is critical with respect to linearization.* Moreover, when $0 \le s < \frac{1}{2}$, we have $\nabla K_{2s} * \rho = \frac{\nabla}{\Lambda}\Lambda^{1-2s}\rho$ for any Schwartz function ρ, which indicates that we have nonlocal property given by Riesz transforms and $1-2s$ order derivative; when $s = \frac{1}{2}$, we only have nonlocal property given by Riesz transforms; when $\frac{1}{2} < s \le \frac{N}{2}$, we have nonlocal properties given by Riesz transforms and Riesz potential Λ^{1-2s} $(1-2s<0)$. As a consequence, $\frac{1}{2}$ *is critical with respect to nonlocal property*, i.e. the bigger s is, the stronger nonlocal property we have.

1.3. Mild solution and scaling argument

In this subsection, we first introduce the definition of mild solution to system (1.1) with initial value $\rho_0(x) = \rho(x,t)|_{t=0}$.

Mild solution Plugging $u\rho = -\mu_{en}\nabla\rho - \mu_{in}\rho\nabla K_{2s} * \rho$ into (1.1) yields

$$\rho_t - \mu_{en}\Delta\rho = \mu_{in}\nabla\cdot(\rho\nabla K_{2s}*\rho). \tag{GD}$$

For any given ρ_0, we get an equivalent integral equation

$$\rho(t) = e^{\mu_{en}t\Delta}\rho_0 + \mu_{in}\int_0^t e^{\mu_{en}(t-\tau)\Delta}\nabla\cdot\big(\rho(\tau)(\nabla K_{2s}*\rho)(\tau)\big)d\tau. \tag{IGD}$$

We call ρ a *mild solution* to (GD) with initial ρ_0 if ρ solves (IGD) in certain function space.

Scaling Formally, the second term on the left hand side of (1.6) gives

$$\int_{\mathbb{R}^N} \rho\, K_{2s}*\rho\, dx \overset{Plancherel's\ identity}{=\!=\!=} \int_{\mathbb{R}^N} |K_s*\rho|^2 dx. \tag{1.12}$$

Therefore it seems quite natural to assume that[3]

$$(K_s*\rho)(x,t) \in L^\infty(0,\infty; L^2(\mathbb{R}^N)) \subset L^2_{loc}(\mathbb{R}^N \times \mathbb{R}_+) \tag{1.13}$$

in the energy framework if $-\int \rho|u|^2dx \le 0$. We can check that it is true for any $0 \le s < \frac{1}{2}$ and $|\rho - \bar\rho_0| < \frac{1}{2}\bar\rho_0$. Indeed, integrating the left hand side of (1.7) with respect to time variable from 0 to t yields

$$0 \le \int_{\mathbb{R}^N} |(K_s * \rho)(t,x)|^2dx \le C(\mu_{en}, \mu_{in}, \rho_0, N), \tag{1.14}$$

where $C(\mu_{en}, \mu_{in}, \rho_0, N, s)$ is a positive constant depending on μ_{en}, μ_{in}, N, s and ρ_0. Meanwhile, taking scaling into consideration, we observe that (1.1) is invariant under the following transformation:

$$\rho(x,t) \mapsto \rho_\lambda(x,t) = \lambda^{2s}\rho(\lambda x, \lambda^2 t) \ \text{ for } \ \lambda > 0. \tag{1.15}$$

As a consequence of (1.12)–(1.15), we have two scale and translation invariant versions of L^2-boundedness:

$$\frac{1}{r^{N-2s}} \int_{B(x;r)} |(K_s*\rho)(y,t)|^2dy, \tag{1.16}$$

$$\frac{1}{r^{N+2-2s}} \int_{Q(x,t;r)} |(K_s*\rho)(y,t)|^2dydt. \tag{1.17}$$

Denote the initial data space as the set of all tempered distributions ρ_0 such that the convolution of $K_{s+1} * \nabla\rho_0$ and heat kernel $G_{\sqrt{t}}$ satisfy

$$\sup_{r>0, x\in\mathbb{R}^N} \frac{1}{r^{N+2-2s}} \int_{Q(x,t;r)} |(G_{\sqrt{t}} * K_{s+1} * \nabla\rho_0)(y)|^2dydt < \infty. \tag{1.18}$$

This space of ρ_0 satisfying (1.18) is BMO^{-2} for $s = 1$, and $\dot{B}^{-2s}_{\infty,\infty}$ for $0 < s < 1$ (see Definitions 1.4 and 1.6 and Lemma 2.5 below). Noticing that (1.14) and (1.16) coincide when $s = \frac{N}{2}$ and $r = \infty$. However, when $s = \frac{N}{2} \ge 1$, linearization argument is not applied. It seems difficult to prove $\int \rho|u|^2dx \ge 0$. Thus we are unable to get estimate (1.14). Later on, we only focus on the mild solution since it seems difficult to apply the a priori energy estimate.

Next we recall some recent results about the Keller–Segel system/two component Keller–Segel system and Poisson–Nernst–Planck system. As for Keller–Segel system, Biler et al. [4] studied its Cauchy problem for initial data $\rho_0 \in \mathcal{PM}^{N-2}$ with $N \ge 4$ and

$$\mathcal{PM}^{N-2} = \{f \in \mathcal{S}' \,|\, \widehat{f} \in L^1_{\text{loc}},\ \|f\|_{\mathcal{PM}^{N-2}} = \text{esssup}_\xi |\xi|^{N-2}|\widehat{f}(\xi)| < \infty\},$$

[3] It is clear that $L^p(0,\infty; L^q(\mathbb{R}^N)) \subset L^2_{loc}(\mathbb{R}^N \times \mathbb{R}_+)$ for any $p, q \ge 2$.

Table 1
Some relations between energy approach and semigroup method.

<table>
<tr><th colspan="3">Energy Approach</th><th colspan="4">Semigroup Method</th></tr>
<tr><td colspan="3">One of the most important term in Energy form is $\int \vert K_s * \rho\vert^2 dx$</td><td colspan="4">One of the most important part of the bilinear interaction term is $\nabla K_{2s} * \rho$</td></tr>
<tr><td rowspan="4">$0 \le s < \frac{N}{2}$</td><td rowspan="4">Initial data space is $\dot{H}^{-s}$; not scaling invariant</td><td rowspan="4">$K_s * \rho$ is well defined for any Schwartz function ρ</td><td>$0 \le s < \frac{1}{2}$</td><td rowspan="2">Initial data space is $\dot{B}^{-2s}_{\infty,q}$ for any $q \ge 1$ and is scaling invariant</td><td rowspan="5">$\nabla K_{2s} * \rho$ is well defined for any Schwartz function ρ</td><td>$\nabla K_{2s} * (\rho + 1) = \nabla K_{2s} * \rho$ and $\Delta K_{2s} * (\rho + 1) = \Delta K_{2s} * \rho$ in distributional sense</td></tr>
<tr><td>$\frac{1}{2} \le s < 1$</td><td>$\nabla K_{2s} * (\rho + 1) = \nabla K_{2s} * \rho$ in distribution modulo polynomial sense; $\Delta K_{2s} * (\rho + 1) = \Delta K_{2s} * \rho$ in distributional sense</td></tr>
<tr><td>$s = 1$</td><td>Initial data space is BMO^{-2} and is scaling invariant</td><td rowspan="3">$\nabla K_{2s} * (\rho + 1) = \nabla K_{2s} * \rho$ and $\Delta K_{2s} * (\rho + 1) = \Delta K_{2s} * \rho$ in distribution modulo polynomial sense.</td></tr>
<tr><td rowspan="2">$1 < s \le \frac{N}{2}$</td><td rowspan="2">Initial data space is $\dot{B}^{-1-s}_{\frac{N}{s-1},q}$ for any $q \ge 1$ and is scaling invariant</td></tr>
<tr><td>$s = \frac{N}{2}$</td><td>Initial data space is $\dot{H}^{-N/2}$; scaling invariant</td><td>$K_s * \rho$ is not well defined for $\rho = e^{-x^2}$</td></tr>
</table>

Corrias et al. [8] established the global well-posedness of the $\rho_0 \in L^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ data problem with only small $L^{\frac{N}{2}}(\mathbb{R}^N)$-norm ($N \ge 2$), and Kozono–Sugiyama [18] investigated both global solution for $\rho_0 \in L^1(\mathbb{R}^2)$ and the blow-up phenomenon. Recently, Iwabuchi [14] proved existence of solution to the Keller–Segel system in $\dot{B}^{N/p-2}_{p,\infty}$ with $N/2 < p < \infty$ and $p \ge 1$, and also in $\dot{\mathcal{B}}^{-2}_2$ (a subspace of BMO^{-2}, see Subsection 3.4 below).

For the two-components Keller–Segel system and the Poisson–Nernst–Planck system, we refer readers to [12,23,22,24,27,30,32,33] to see more information about the existence, uniqueness and asymptotic behaviors of the solutions. Generally speaking, scaling invariant space with lower regular index is bigger. Hence it is worth pointing out that Zhao et al. [33] proved global well-posedness of the two-components Poisson–Nernst–Planck system in $\dot{B}^s_{p,\infty}$ with $s > -3/2$ and $p = N/(s+2)$, which is the first result that work for regular index below -1 of this model. Recently, Deng and Li [9] extended Zhao et al.'s work to critical case, established ill-posedness of the two-components Poisson–Nernst–Planck system in $\dot{B}^{-3/2}_{2N,q}$ for $N = 2$ and $q > 2$, and showed that the regular index $s = -3/2$ is optimal.

Before ending this subsection, we give Table 1 concerned with the relations between energy approach and semigroup method (mild solution).

Let us end this subsection with our main results. The initial value problem of system (1.1) is well-posed: in the largest scaling invariant space $\dot{B}^{-2s}_{\infty,\infty}$ for any $0 < s < 1$, see Theorem 1.9 and Remark 1.10 below; in the largest scaling invariant space BMO^{-2s} for $s = 1$, see Theorem 1.11 below; in the scaling invariant space $\dot{B}^{-2s}_{\infty,1}$ for $s = 0$, where we do not know whether it is the largest or not; in the scaling invariant space $\dot{B}^{-3}_{4,2}$ for $N = 4$ and $s = \frac{N}{2} > 1$ where integrability can not be ∞, i.e. $\dot{B}^{-2s}_{\infty,q}$ is a the proper choice.

It should be an interesting problem whether system (1.1) is globally well-posed in the homogeneous Sobolev space $\dot{H}^{-\frac{N}{2}}$ for arbitrary large initial data, in Besov space $\dot{B}^{-1-s}_{N/(s-1),q}$ for $1 < s \le \frac{N}{2}$ and $1 \le q \le \infty$, or in Triebel–Lizorkin space $\dot{F}^{-1-s}_{N/(s-1),q}$ for $1 < s \le \frac{N}{2}$ and $1 \le q \le \infty$.

1.4. Notations and definitions

In this subsection, we list the notations which will be used throughout this paper as follows:

N	space dimension and $N \in \{2, 3, 4, \cdots\}$,
$\mathbb{R}_+$, $\mathbb{N}$, $\mathbb{Z}_+$	$\mathbb{R}_+ = (0, \infty)$, $\mathbb{N} = \{1, 2, 3, \cdots, \}$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$,
$(\mathcal{F}v)(\xi)$ or $\hat{v}(\xi)$	Fourier transformation of v with respect to x,
$(\mathcal{F}^{-1}\hat{v})(x)$	inverse Fourier transformation of $\hat{v}$ with respect to ξ,
$K_s(x)$	kernel of Riesz operator $(-\Delta)^{-\frac{s}{2}}$,
$f^{(s)}$	$f^{(s)} := K_s * f$ for any function f and $0 < s \le 1$,
$G_{\sqrt{t}}(x)$	kernel of heat semigroup $e^{t\Delta}$, i.e. $(2\pi t)^{-\frac{N}{2}} \exp\{-\frac{\lvert x\rvert^2}{4t}\}$,
$B(x; r)$	space ball centered at $x \in \mathbb{R}^N$ of radius r,
$Q(x, t; r)$	parabolic ball $Q(x, t; r) = B(x; r) \times (0, r^2[$,
$[0, 1]^N$	N-dimensional unit cube,
$\dot{B}^{\sigma}_{\infty,q}$	homogeneous Besov space for $\sigma \in \mathbb{R}$ and $q \in [1, \infty]$,
BMO	bounded mean oscillation space and $BMO = \dot{F}^0_{\infty,2}$,
BMO^{σ}	$-\sigma$th order derivative of BMO space and $BMO^{\sigma} = \dot{F}^{\sigma}_{\infty,2}$,
$C_{a,b,\cdots}$	positive constant depending on a, b, $\cdots$,
$A \lesssim B$, $A \sim B$	$A \lesssim B \Leftrightarrow A \le C_{N,s,\mu_{en},\mu_{in}} B$ and $A \sim B \Leftrightarrow A \lesssim B \lesssim A$,
$\mathcal{S}$ and $\mathcal{S}'$	Schwartz function space $\mathcal{S}(\mathbb{R}^N)$ and tempered distribution space $\mathcal{S}'(\mathbb{R}^N)$,
L^q_x, L^p_t and L^r_ξ	$L^q_x = L^q(\mathbb{R}^N)$, $L^p_t = L^p(\mathbb{R}_+)$ and $L^r_\xi = L^r(\mathbb{R}^N)$,
Λ and $\mathcal{R}_j$	Riesz potential $\Lambda = \sqrt{-\Delta}$ and Riesz transform $\mathcal{R}_j = \frac{\partial_j}{\Lambda}$.

Next we define the homogeneous Littlewood–Paley decomposition. Assume that $\psi \in \mathcal{S}$, $0 \le \psi \le 1$, $\psi \equiv 1$ on $\{\xi \in \mathbb{R}^N / \ \frac{5}{6} \le \lvert\xi\rvert \le \frac{4}{3}\}$ and $\operatorname{supp} \psi \subset \{\xi \in \mathbb{R}^N / \ \frac{1}{2} < \lvert\xi\rvert < 2\}$ with

$$\sum_{k\in\mathbb{Z}} \psi(2^{-k}\xi) \equiv 1 \quad \text{for any } \xi \in \mathbb{R}^N \backslash \{0\}.$$

Let $\Delta_k v = \mathcal{F}^{-1}\psi(2^{-k}\xi)\hat{v}(\xi)$, $\widetilde{\Delta}_k = \Delta_{k-3} + \cdots + \Delta_{k+3}$, $\widehat{\widetilde{\Delta}_k v}(\xi) = \widetilde{\psi}(2^{-k}\xi)\widehat{v}(\xi)$ and

$$P_{\leq k} v = \sum_{j\leq k} \Delta_j v = \mathcal{F}^{-1}\big(p(2^{-k}\xi)\hat{v}(\xi)\big),$$

where $\operatorname{supp} p \subset \{\xi \in \mathbb{R}^N / \ |\xi| \leq 2\}$. Then for any $\ell \in \{-2, -1, 0, 1, 2\}$ we get

$$\Delta_k u P_{\leq k-3} v = \widetilde{\Delta}_k(\Delta_k u P_{\leq k-3} v), \quad \Delta_k u \Delta_{k-\ell} v = P_{\leq k+3}(\Delta_k u \Delta_{k-\ell} v)$$

and the following decomposition of product uv, i.e.

$$\begin{aligned} uv &= \sum_{k\in\mathbb{Z}} \widetilde{\Delta}_k(\Delta_k u P_{\leq k-3} v) + \sum_{k\in\mathbb{Z}} \widetilde{\Delta}_k(P_{\leq k-3} u \Delta_k v) \\ &\quad + \sum_{k\in\mathbb{Z}} \sum_{|\ell|\leq 2} \widetilde{\Delta}_k(\Delta_k u \Delta_{k-\ell} v) + \sum_{k\in\mathbb{Z}} \sum_{|\ell|\leq 2} P_{\leq k-4}(\Delta_k u \Delta_{k-\ell} v) \\ &:= \Pi_h^{hl}(u,v) + \Pi_h^{lh}(u,v) + \Pi_h^{hh}(u,v) + \Pi_l^{hh}(u,v), \end{aligned} \tag{Π}$$

where Π_h^{hl} is *high–low to high* interaction (similar conventions are applied).

For any $\phi(x) \in \mathcal{S}$ there exists a positive constant $C_{N,\phi}$ such that

$$\sum_{k\in\mathbb{Z}^N} \sup_{x\in k+[0,1]^N} |\phi(x)| \leq \sum_{k\in\mathbb{Z}^N} \frac{\sup_{x\in k+[0,1]^N} (1+|x|)^{N+1}|\phi(x)|}{(1+|k|)^{N+1}} < C_{N,\phi}.$$

We define the space of functions satisfying the above property by L^1_{sup}.

Definition 1.1. For any $N \in \mathbb{N} \cap [2, \infty)$, we define L^1_{sup} as the space of tempered distributions v such that

$$\|v\|_{L^1_{sup}} = \sum_{k\in\mathbb{Z}^N} \sup_{r>0,\ \frac{x}{r}\in k+[0,1]^N} |v(x)| < \infty. \tag{1.19}$$

Remark 1.2. (1.19) yields $L^1_{sup} \subset L^1_x$. Moreover, for any $t > 0$, $G_{\sqrt{t}}(x) \in L^1_{sup}$. Similarly, one can check that for any $\phi \in \mathcal{S}$ and $r > 0$, we get $\frac{1}{r^N}\phi(\frac{x}{r}) \in L^1_{sup}$.

Next we define the *uniformly local space* L^p_{uloc}.

Definition 1.3. For any $N \in \mathbb{N} \cap [2, \infty)$ and $p \in [1, \infty)$, we define L^p_{uloc} as the *uniformly local space* of distributions $u(x,t)$ on $\mathbb{R}^N \times \mathbb{R}_+$ so that

$$\|u\|_{L^p_{uloc}} = \sup_{x\in\mathbb{R}^N, r>0} \Big(\frac{1}{r^N} \int_{Q(x,t;r)} |u(y,t)|^p dy dt\Big)^{\frac{1}{p}} < \infty. \tag{1.20}$$

Below is an equivalent characterization of BMO^{-2s} via Carleson measure.

Definition 1.4. For any $N \in \mathbb{N} \cap [2, \infty)$ and $s \in [0, 1]$, we defined BMO^{-2s} to be the space of all tempered distributions v such that

$$\|v\|_{BMO^{-2s}} = \|w\|_{L^2_{uloc}} < \infty, \tag{1.21}$$

where $w(x, t) := e^{t\Delta}\nabla K_{2s} * v(x)$.

Remark 1.5. Recall from [17] that $v \in BMO \iff e^{t\Delta}\nabla v \in L^2_{uloc}$. Similarly, we get $K_{2s} * v \in BMO \iff e^{t\Delta}\nabla K_{2s} * v \in L^2_{uloc}$. Let $h = K_{2s} * v$. Then $v \in BMO^{-2s} \Leftrightarrow h \in BMO$ and $v = (-\Delta)^s h$. Thus any given BMO^{-2s} function can be written as the $2s$ order derivative of a BMO function. In particular, by the boundedness of Riesz transforms in the homogeneous Triebel–Lizorkin spaces, BMO^{-2s} ($s = \frac{1}{2}$) coincides with the BMO^{-1} defined in [17].

At last, we recall the definition of Besov type spaces.

Definition 1.6. For $\sigma \in \mathbb{R}$ and $p, q \in [1, \infty]$, we defined $\dot{B}^\sigma_{p,q}$ to be the space of tempered distributions $v(x)$ such that

$$\|v\|_{\dot{B}^\sigma_{p,q}} = \Big(\sum_{k\in\mathbb{Z}} 2^{\sigma kq}\|\Delta_k v\|^q_{L^p_x}\Big)^{\frac{1}{q}} < \infty.$$

Similarly, for any $\sigma \in \mathbb{R}$ and $(p, q, r) \in [1, \infty]^3$, we defined $\widetilde{L}^r_t(\dot{B}^\sigma_{p,q})$ to be the space of tempered distribution $u(x, t)$ such that

$$\|u\|_{\widetilde{L}^r_t(\dot{B}^\sigma_{p,q})} = \Big(\sum_{k\in\mathbb{Z}} 2^{\sigma kq}\|\Delta_k u\|^q_{L^r_t L^p_x}\Big)^{\frac{1}{q}} < \infty.$$

1.5. Main results

In this subsection, we state the results on the existence and uniqueness of the mild solution of the system (IGD) with initial data ρ_0 belonging to: BMO^{-2s} for $s = 1$; $\dot{B}^{-2s}_{\infty,1}$ for $s = 0$; $\dot{B}^{-2s}_{\infty,q}$ for $(s, q) \in (0, 1) \times [1, \infty]$; and $\dot{B}^{-3}_{4,2}$ for $(s, N) = (2, 4)$.

Theorem 1.7. *Let $N \in \mathbb{N} \cap [2, \infty)$ and $s = 0$. Then there exists $\varepsilon > 0$ such that the general diffusion system (GD) with initial data $\rho_0 \in \dot{B}^0_{\infty,1}$ and $\|\rho_0\|_{\dot{B}^0_{\infty,1}} < \varepsilon$ has a unique global mild solution $\rho \in C([0, \infty); \dot{B}^0_{\infty,1})$ satisfying*

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1}) \cap \widetilde{L}^2_t(\dot{B}^1_{\infty,1})} < 2c\varepsilon.$$

Remark 1.8. Recall from Definition 1.6 and [10, Lemma 5], it is easy to check that $\dot{B}^0_{\infty,1} \subset BUC$, where BUC is the space of bounded uniformly continuous function. Thus time continuity of the heat semigroup follows immediately.

Theorem 1.9. *Let $d \in \mathbb{N} \cap [2, \infty)$ and $(s, q) \in (0, 1) \times [1, \infty]$. Then there exists $\varepsilon > 0$ so that the general diffusion system (GD) with initial data $\rho_0 \in \dot{B}^{-2s}_{\infty,q}$ and $\|\rho_0\|_{\dot{B}^{-2s}_{\infty,q}} < \varepsilon$ has a unique global mild solution ρ satisfying that for any $1 \leq q < \infty$, $\rho \in C([0, \infty); \dot{B}^{-2s}_{\infty,q})$ and*

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q}) \cap \widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})} < 2c\varepsilon;$$

for $q = \infty$, $\rho \in C_w([0, \infty); \dot{B}^{-2s}_{\infty,q})$ and

$$\|\rho\|_{L^\infty_t(\dot{B}^{-2s}_{\infty,\infty}) \cap \widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,\infty})} < 2c\varepsilon,$$

where $C_w([0, \infty); \dot{B}^{-2s}_{\infty,\infty})$ denotes the space of all $\dot{B}^{-2s}_{\infty,\infty}$ valued weakly continuous functions $\rho(t)$ defined for $t \in [0, \infty)$.

Remark 1.10. For $N \geq 2$ and $s = \frac{1}{2}$, we can prove the following results:

i) there exists $\varepsilon > 0$ so that (GD) with $\rho_0 \in BMO^{-1}$ and $\|\rho_0\|_{BMO^{-1}} < \varepsilon$ has a unique global mild solution ρ satisfying

$$\sup_{t>0} t^{\frac{1}{2}} \|\rho\|_{L^\infty_x} + \sup_{t>0} t^{\frac{1}{2}} \|\mathcal{R}_j \rho\|_{L^\infty_x} + \|\rho\|_{L^2_{uloc}} + \|\mathcal{R}_j \rho\|_{L^2_{uloc}} < 2c\varepsilon;$$

ii) there exists $\varepsilon > 0$ so that (GD) with $\rho_0 \in \dot{B}^{-1}_{\infty,\infty}$ and $\|\rho_0\|_{\dot{B}^{-1}_{\infty,\infty}} < \varepsilon$ has a unique solution $\rho \in C_w([0, \infty); \dot{B}^{-1}_{\infty,\infty})$, $\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-1}_{\infty,\infty}) \cap \widetilde{L}^2_t(\dot{B}^0_{\infty,\infty})} < 2c\varepsilon$;

iii) in general, for any $s \in [0, 1]$, we have (see Lemma 2.1 below)

$$\rho \nabla K_{2s} * \rho = \Delta T_{s,1}(\rho, \rho) + \nabla T_{s,2}(\rho, \rho) + \nabla \cdot T_{s,3}(\rho, \rho).$$

Theorem 1.11. *Let $N \in \mathbb{N} \cap [2, \infty)$ and $s = 1$. Then there exists $\varepsilon > 0$ such that (GD) with initial data $\rho_0 \in BMO^{-2}$ and $\|\rho_0\|_{BMO^{-2}} < \varepsilon$ has a unique global mild solution ρ satisfying*

$$\sup_{t>0} t^{\frac{1}{2}} \left\|\nabla K_2 * \rho\right\|_{L^\infty_x} + \left\|\nabla K_2 * \rho\right\|_{L^2_{uloc}} < 2c\varepsilon.$$

Next we consider one special case for $s > 1$ in 4-dimensional space.

Theorem 1.12. *Let $N = 4$ and $s = 2$. There exists $\varepsilon > 0$ such that (GD) with $\rho_0 \in \dot{B}^{-3}_{4,2}$ and $\|\rho_0\|_{\dot{B}^{-3}_{4,2}} < \varepsilon$ has a unique global mild solution ρ satisfying*

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-3}_{4,2}) \cap \widetilde{L}^2_t(\dot{B}^{-2}_{4,2})} \leq 2c\varepsilon.$$

Remark 1.13. Theorem 1.12 is only one of the endpoint cases for $N \geq 3$ and $1 < s \leq \frac{N}{2}$. It seems that our arguments can not be extended to the general case, especially in $\dot{B}^{-1-s}_{\frac{N}{s-1},q}$

for $1 < s \le \frac{N}{2}$, $4 < \frac{N}{s-1}$, $1 \le q$. It is worth mentioning that it is difficult to get the a priori estimate by using (1.6) since $\int_{\mathbb{R}^4} |K_2 * \rho| dx$ is not well-defined for functions and $\int_{\mathbb{R}^4} \rho |u|^2 dx$ is necessarily nonnegative.

Plan of the paper. In Section 2, we do some preliminary arguments. In Section 3, we give the proof of our main results.

2. Preliminaries

From now on, we assume that $0 \le s \le 1$. Notice that the following bilinear operator

$$B(\rho,\rho) = \mu_{in} \int_0^t e^{\mu_{en}(t-\tau)\Delta} \nabla \cdot \big(\rho \nabla K_{2s} * \rho\big)(\tau) d\tau \tag{O_b}$$

is the solution to the following equation with 0 initial data, i.e.

$$\begin{cases} \rho_t - \mu_{en}\Delta\rho = \mu_{in}\nabla\cdot\big(\rho\nabla K_{2s} * \rho\big), \\ \rho|_{t=0} = 0. \end{cases} \tag{2.1}$$

In order to estimate $B(\rho,\rho)$, the key point is to take advantage of the *potential cancellation property* of $\rho\nabla K_{2s} * \rho$.

2.1. Bilinear pseudodifferential calculus

In this subsection, we study $\rho\nabla K_{2s} * \rho$ via Fourier analysis tools, i.e.

$$\rho\nabla K_{2s} * \rho = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \sigma_s(\xi,\zeta)\,\widehat{\rho}(\xi)\widehat{\rho}(\zeta) e^{ix\cdot(\xi+\zeta)} d\zeta d\xi, \tag{2.2}$$

where

$$\sigma_s(\xi,\zeta) = c_N\,(i\zeta|\xi|^{2s} + i\xi|\zeta|^{2s})|\xi|^{-2s}|\zeta|^{-2s}. \tag{2.3}$$

To deal with (2.2), we recall some related works on bilinear/multilinear pseudodifferential calculus, see [3,20,25,26] and references therein. Recall that the bilinear operator

$$T_m(f,g)(x) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} m(\xi,\zeta)\widehat{f}(\xi)\widehat{g}(\zeta) e^{ix\cdot(\xi+\zeta)} d\zeta d\xi \tag{2.4}$$

is defined in [26] for any $f, g \in \mathcal{S}$.

An interesting example of a similar flavor in nonlinear PDEs is given by Kato–Ponce [15]. If $f, g \in \mathcal{S}$ and $\widehat{\Lambda^a f}(\cdot) = |\cdot|^a \widehat{f}(\cdot)$ with $a > 0$, then

$$\|\Lambda^a(fg)\|_{L^r_x} \lesssim \|\Lambda^a f\|_{L^p_x}\|g\|_{L^q_x} + \|f\|_{L^p_x}\|\Lambda^a g\|_{L^q_x} \tag{2.5}$$

for any $1 < p, q \leq \infty$, $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Recently, Bourgain–Li [5] extended (2.5) to endpoint case, i.e. $r = p = q = \infty$ and $a > 0$.

Roughly speaking, if f oscillates more rapidly than g, then g is essentially constant with respect to f, and so $\Lambda^a(fg)$ behaves like $(\Lambda^a f)g$. Similarly, one expects $\Lambda^a(fg)$ to be like $f(\Lambda^a g)$ if g oscillates more rapidly than f. This is why there are two terms on the right hand side of (2.5). It is worth mentioning that (2.5) is not true for $a < 0$ due to the counterexample

$$(f, g) = (\cos n_0 x_1, \cos(n_0 - 1)x_1) \text{ for large } n_0,$$

with $a = -2$, $r = p = q = \infty$ and $N \geq 2$.

Recall the definition of $m(\xi, \zeta)$ in (2.4), if we additionally assume that $m(\xi, \zeta) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is bounded, smooth away from $\{\xi = 0\} \cup \{\zeta = 0\}$ and satisfies the Marcinkiewicz–Mikhlin–Hörmander type condition

$$\left|\partial_\xi^\alpha \partial_\zeta^\beta m(\xi, \zeta)\right| \lesssim \frac{1}{|\xi|^{|\alpha|}|\zeta|^{|\beta|}} \tag{2.6}$$

for sufficiently many multi-indices[4] $\alpha, \beta \in \mathbb{Z}_+^N$, then Muscalu, Pipher, Tao and Thiele established the following theorem, see Theorem 1.3 of [26].

Theorem 1.3. *The bilinear operator T_m defined in (2.4) maps $L_x^{p_1} \times L_x^{p_2} \mapsto L_x^p$ boundedly as long as $1 < p_1, p_2 \leq \infty$, $0 < p < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.*

We can slightly generalize the above Theorem 1.3. Define

$$T_{m_{a,b}}(f, g)(x) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} m_{a,b}(\xi, \zeta)\widehat{f}(\xi)\widehat{g}(\zeta)e^{ix\cdot(\xi+\zeta)}d\zeta d\xi,$$

where $a, b \geq 0$. Additionally, if $m_{a,b}(\xi, \zeta) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ is smooth away from the subspace $\{\xi = 0\} \cup \{\zeta = 0\}$ and satisfies

$$\left|\partial_\xi^\alpha \partial_\zeta^\beta m_{a,b}(\xi, \zeta)\right| \lesssim \frac{1}{|\xi|^{a+|\alpha|}|\zeta|^{b+|\beta|}}$$

for sufficiently many multi-indices α and β. For $1 < p < \infty$ and $\sigma \leq 0$, define

$$\dot{\mathcal{L}}_\sigma^p := \left\{v \in \mathcal{S}'(\mathbb{R}^N) /\ \|\Lambda^\sigma v\|_{L_x^p} < \infty\right\},$$

then by direct application of Theorem 1.3 we have the following results:

[4] $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $|\alpha| = \alpha_1 + \cdots \alpha_N$.

Theorem 1.3′. *The bilinear operator $T_{m_{a,b}}$ maps $\dot{\mathcal{L}}^{p_1}_{-a} \times \dot{\mathcal{L}}^{p_2}_{-b} \mapsto L^p_x$ boundedly as long as $0 \le a,\, b < \infty$, $1 < p_1, p_2 < \infty$, $0 < p < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.*

Recall the definition of the bilinear symbol $\sigma_s(\xi,\zeta)$ defined in (2.2), we observe that $\sigma_s(\xi,\zeta)$ is symmetric and away from $\{\xi = 0\} \cup \{\zeta = 0\}$,

$$\left|\partial_\xi^\alpha \partial_\zeta^\beta \sigma_s(\xi,\zeta)\right| \lesssim \frac{1}{|\xi|^{2s-1+|\alpha|}|\zeta|^{|\beta|}} + \frac{1}{|\xi|^{|\alpha|}|\zeta|^{2s-1+|\beta|}}.$$

However, in the endpoint case, i.e. $p_1 = p_2 = p = \infty$, Theorem 1.3′ does not apply. Therefore, we might need to make full use of the symmetric and the cancellation properties of the bilinear multiplier $\sigma_s(\xi,\zeta)$. Precisely speaking, we will split $\rho\nabla K_{2s} * \rho$ into three pieces, i.e.

$$\rho\nabla K_{2s} * \rho = \Delta T_{s,1}(\rho,\rho) + \nabla T_{s,2}(\rho,\rho) + \nabla \cdot T_{s,3}(\rho,\rho), \tag{2.7}$$

where all $T_{s,j}(\rho,\rho)$ $(j = 1,2,3)$ can be well controlled.

It is worth mentioning that the *identity* (2.7) plays a crucial role in the study of the mild solution of (IGD).

In the following lemma we shall give the detail proof of identity (2.7).

Lemma 2.1. *Let $\widehat{\Delta_k^\mu \rho}(\xi) = e^{i\mu\cdot 2^{-k}\xi}\psi(2^{-k}\xi)\widehat{\rho}(\xi)$, $\widehat{\Delta_{k-\ell}^\nu \rho}(\zeta) = e^{i\nu\cdot 2^{-k}\zeta}\psi(2^{\ell-k}\zeta)\widehat{\rho}(\zeta)$. Then for any $0 \le s \le 1$ we get*

$$\rho\nabla K_{2s} * \rho = \Delta T_{s,1}(\rho,\rho) + \nabla T_{s,2}(\rho,\rho) + \nabla \cdot T_{s,3}(\rho,\rho), \tag{2.8}$$

where

$$\begin{cases} T_{s,1}(\rho,\rho) = -\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} P_{\le k-4}\big(\nabla K_{2+2s} * \Delta_k^\mu \rho\, \Delta_{k+\ell}^\nu \rho\big) r^1_{\ell 1}(\mu,\nu) d\mu d\nu \\ \qquad\qquad -\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} \widetilde{\Delta}_k\big(\nabla K_{2s} * \Delta_k^\mu \rho\, K_2 * \Delta_{k+\ell}^\nu \rho\big) r^1_{\ell 2}(\mu,\nu) d\mu d\nu \\ \qquad\qquad -\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} \widetilde{\Delta}_k\big(K_2 * \Delta_k^\mu \rho\, \nabla K_{2s} * \Delta_{k+\ell}^\nu \rho\big) r^2_{\ell 2}(\mu,\nu) d\mu d\nu \\ \qquad\qquad - K_2 * \Pi_h^{hl}(\rho, \nabla K_{2s} * \rho) - K_2 * \Pi_h^{lh}(\rho, \nabla K_{2s} * \rho), \\ T_{s,2}(\rho,\rho) = \displaystyle\sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} P_{\le k-4}(\Delta_k \rho\, K_{2s} * \Delta_{k+\ell}\rho), \\ T_{s,3}(\rho,\rho) = \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} P_{\le k-4}\big(\nabla\nabla K_{2+2s} * \Delta_k^\mu \rho\, \Delta_{k+\ell}^\nu \rho\big) r^2_{\ell 1}(\mu,\nu) d\mu d\nu. \end{cases}$$

In particular, for $s = 0$, we get $T_{s,1}(\rho,\rho) = T_{s,3}(\rho,\rho) = 0$ and $T_{s,2} = \frac{1}{2}\rho^2$, i.e.

$$\rho\nabla\rho = \nabla(\frac{1}{2}\rho^2); \tag{2.9}$$

for $s = 1$, *we get* $T_{s,1}(\rho,\rho) = 0$, $T_{s,2}(\rho,\rho) = \frac{1}{2}|\nabla K_2 * \rho|^2$ *and* $T_{s,3}(\rho,\rho) = -\nabla K_2 * \rho \otimes \nabla K_2 * \rho$, *i.e.*

$$\rho\nabla K_2 * \rho = -\nabla\cdot\big(\nabla K_2 * \rho \otimes \nabla K_2 * \rho\big) + \nabla\big(\frac{1}{2}|\nabla K_2 * \rho|^2\big). \tag{2.10}$$

Proof. Applying decomposition (II) to $\rho\nabla K_{2s} * \rho$ yields

$$\begin{aligned}\rho\nabla K_{2s} * \rho =& \Pi_h^{hl}(\rho, \nabla K_{2s} * \rho) + \Pi_h^{lh}(\rho, \nabla K_{2s} * \rho)\\ &+ \Pi_h^{hh}(\rho, \nabla K_{2s} * \rho) + \Pi_l^{hh}(\rho, \nabla K_{2s} * \rho).\end{aligned} \tag{2.11}$$

It suffices to rewrite $\Pi_h^{hh}(\rho, \nabla K_{2s} * \rho) + \Pi_l^{hh}(\rho, \nabla K_{2s} * \rho)$ since

$$\begin{aligned}\Pi_h^{hl}(\rho, \nabla K_{2s} * \rho) &= -\Delta K_2 * \Pi_h^{hl}(\rho, \nabla K_{2s} * \rho),\\ \Pi_h^{lh}(\rho, \nabla K_{2s} * \rho) &= -\Delta K_2 * \Pi_h^{lh}(\rho, \nabla K_{2s} * \rho).\end{aligned} \tag{2.12}$$

For the sake of simplicity, we shall denote $p_k(\cdot) = p(2^{-k}\cdot)$, $p_0(\cdot) = p(\cdot)$, $\psi_k(\cdot) = \psi(2^{-k}\cdot)$, $\psi_0(\cdot) = \psi(\cdot)$, $\widetilde{\psi}_k(\cdot) = \psi_{k-3}(\cdot) + \cdots \psi_{k+3}(\cdot)$ and $\widetilde{\psi}(\cdot) = \psi_{-3}(\cdot) + \cdots \psi_3(\cdot)$, respectively.

Since ∇ commutes with Δ_k, i.e. $\Delta_k \nabla K_{2s} * \rho = \nabla K_{2s} * \Delta_k\rho$, we have

$$\begin{aligned}\Pi_h^{hh}(\rho, \nabla K_{2s} * \rho) + \Pi_l^{hh}(\rho, \nabla K_{2s} * \rho) &= \sum_{k\in\mathbb{Z}}\sum_{\ell=-2}^{2} \Delta_k\rho\, \Delta_{k-\ell}\nabla K_{2s} * \rho\\ &= \sum_{k\in\mathbb{Z}} \big(\Delta_k\rho\, \nabla K_{2s} * \Delta_{k+2}\rho + \Delta_{k+2}\rho\, \nabla K_{2s} * \Delta_k\rho\big)\\ &\quad + \sum_{k\in\mathbb{Z}} \big(\Delta_k\rho\, \nabla K_{2s} * \Delta_{k+1}\rho + \Delta_{k+1}\rho\, \nabla K_{2s} * \Delta_k\rho\big)\\ &\quad + \sum_{k\in\mathbb{Z}} \frac{1}{2}\big(\Delta_k\rho\, \nabla K_{2s} * \Delta_k\rho + \nabla K_{2s} * \Delta_k\rho\, \Delta_k\rho\big)\\ &:= \mathcal{H}_2(\rho,\rho) + \mathcal{H}_1(\rho,\rho) + \mathcal{H}_0(\rho,\rho).\end{aligned} \tag{2.13}$$

With no loss of generality, we only need to estimate $\mathcal{H}_2(\rho,\rho)$. It is easy to check that

$$\begin{aligned}&\mathcal{H}_2(\rho,\rho)\\ &= c_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{i(\zeta|\xi|^{2s} + \xi|\zeta|^{2s}) p_{k+3}(\xi+\zeta)\, \psi_k(\xi)\psi_{k+2}(\zeta)}{|\xi|^{2s}|\zeta|^{2s}} \widehat{\rho}(\xi)\widehat{\rho}(\zeta)\, e^{ix\cdot(\xi+\zeta)} d\xi d\zeta\\ &= c_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{i(\zeta|\xi|^{2s} + \xi|\zeta|^{2s}) p_{k-4}(\xi+\zeta)\, \psi_k(\xi)\psi_{k+2}(\zeta)}{|\xi|^{2s}|\zeta|^{2s}} \widehat{\rho}(\xi)\widehat{\rho}(\zeta)\, e^{ix\cdot(\xi+\zeta)} d\xi d\zeta\end{aligned}$$

$$+ c_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\widetilde{\psi}_k(\xi+\zeta)\,\psi_k(\xi)\psi_{k+2}(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}\widehat{\rho}(\xi)\widehat{\rho}(\zeta)\,e^{ix\cdot(\xi+\zeta)}d\xi d\zeta$$

$$:= \mathcal{H}_{21}(\rho,\rho) + \mathcal{H}_{22}(\rho,\rho). \tag{2.14}$$

Consider the symbol of $\mathcal{H}_{21}(\rho,\rho)$, i.e.

$$m_k^*(\xi,\zeta) = \frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\,p_{k-4}(\xi+\zeta)\psi_k(\xi)\psi_{k+2}(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}. \tag{2.15}$$

Notice that $m_k^*(\xi,\zeta) = 2^{(1-2s)k}m^*(\frac{\xi}{2^k},\frac{\zeta}{2^k})$ where

$$m^*(\xi,\zeta) = \frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\,p_{-4}(\xi+\zeta)\psi(\xi)\psi_2(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}$$

and $\operatorname{supp} m^* \subset \{(\xi,\zeta)\in\mathbb{R}^N\times\mathbb{R}^N /\ 2^{-1}<|\xi|<2,\ 2<|\zeta|<2^3,\ |\xi+\zeta|<2^{-3}\}$.

Let $h,\eta\in\mathcal{S}$ be such that $h\equiv 1$ on $\operatorname{supp}\psi$ with $\operatorname{supp}h\subset\{\xi /\ \frac{1}{3}<|\xi|<\frac{7}{3}\}$ and $\eta\equiv 1$ on $\operatorname{supp}\psi_2$ with $\operatorname{supp}\eta\subset\{\zeta /\ \frac{1}{3}2^2<|\zeta|<\frac{7}{3}2^2\}$. Then

$$m^*(\xi,\zeta) = \frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\,p_{-4}(\xi+\zeta)\psi(\xi)\psi_2(\zeta)h(\xi)\eta(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}, \tag{2.16}$$

and $\frac{\zeta|\xi|^{2s}+\xi|\zeta|^{2s}}{|\xi|^{2s}|\zeta|^{2s}} = \frac{\xi(|\zeta|^{2s}-|\xi|^{2s})}{|\xi|^{2s}|\zeta|^{2s}} + \frac{\xi+\zeta}{|\zeta|^{2s}}$. Moreover,

$$\begin{aligned}
&\frac{i\xi(|\zeta|^{2s}-|\xi|^{2s})h(\xi)\eta(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}\\
&= \int_0^1 \frac{2s\,i\xi(\xi+\zeta)\cdot\big(\theta(\xi+\zeta)-\xi\big)\big|\theta(\xi+\zeta)-\xi\big|^{2s}}{\big|\theta(\xi+\zeta)-\xi\big|^2|\xi|^{2s}|\zeta|^{2s}}h(\xi)\eta(\zeta)d\theta\\
&= \frac{i(\xi+\zeta)\cdot i\xi\, i\xi}{|\xi|^{2+2s}}\int_0^1 \frac{2s\,\big|\theta(\xi+\zeta)-\xi\big|^{2s}|\xi|^2}{\big|\theta(\xi+\zeta)-\xi\big|^2|\zeta|^{2s}}h(\xi)\eta(\zeta)d\theta\\
&\quad + \frac{i\xi|\xi+\zeta|^2}{|\xi|^{2+2s}}\int_0^1 \frac{2s\big|\theta(\xi+\zeta)-\xi\big|^{2s}|\xi|^2}{\big|\theta(\xi+\zeta)-\xi\big|^2|\zeta|^{2s}}h(\xi)\eta(\zeta)\,\theta d\theta.
\end{aligned} \tag{2.17}$$

Define

$$\tau_{21}^1(\xi,\zeta) = \int_0^1 \frac{2s\big|\theta(\xi+\zeta)-\xi\big|^{2s}|\xi|^2}{\big|\theta(\xi+\zeta)-\xi\big|^2|\zeta|^{2s}}h(\xi)\eta(\zeta)\,\theta d\theta,$$

$$\tau_{21}^2(\xi,\zeta) = \int_0^1 \frac{2s\,\big|\theta(\xi+\zeta)-\xi\big|^{2s}|\xi|^2}{\big|\theta(\xi+\zeta)-\xi\big|^2|\zeta|^{2s}}h(\xi)\eta(\zeta)\,d\theta.$$

Observe that $|\theta(\xi+\zeta)-\xi|\sim|\xi|$ for $(\xi,\zeta)\in \operatorname{supp} h\times \operatorname{supp}\eta$. Thus $\tau_{21}^1,\tau_{21}^2\in\mathcal{S}(\mathbb{R}^N\times\mathbb{R}^N)$ and

$$\tau_{21}^j(\xi,\zeta)=c_N\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} e^{i(\mu\cdot\xi+\nu\cdot\zeta)}r_{21}^j(\mu,\nu)d\mu d\nu \tag{2.18}$$

with $j=1,2$.

It is easy to check that $r_{21}^1,r_{21}^2\in\mathcal{S}(\mathbb{R}^N\times\mathbb{R}^N)$. Plugging (2.17)–(2.18) into (2.16) yields

$$\begin{aligned} m^*(\xi,\zeta)=&c_N\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} i(\xi+\zeta)\cdot p_{-4}(\xi+\zeta)\frac{i\xi i\xi\, e^{i\mu\cdot\xi}\psi(\xi)e^{i\nu\cdot\zeta}\psi_2(\zeta)}{|\xi|^{2s+2}}r_{21}^2(\mu,\nu)d\mu d\nu\\ &+c_N\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}|\xi+\zeta|^2p_{-4}(\xi+\zeta)\,\frac{i\xi e^{i\mu\cdot\xi}\psi(\xi)e^{i\nu\cdot\zeta}\psi_2(\zeta)}{|\xi|^{2s+2}}r_{21}^1(\mu,\nu)d\mu d\nu\\ &+i(\xi+\zeta)\,p_{-4}(\xi+\zeta)\frac{\psi(\xi)\psi_2(\zeta)}{|\zeta|^{2s}}. \end{aligned}\tag{2.19}$$

Consider the symbol of $\mathcal{H}_{22}(\rho,\rho)$, i.e.

$$m_k^{\#}(\xi,\zeta)=\frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\,\widetilde{\psi}_k(\xi+\zeta)\psi_k(\xi)\psi_{k+2}(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}. \tag{2.20}$$

Notice that $m_k^{\#}(\xi,\zeta)=2^{(1-2s)k}m^{\#}(\frac{\xi}{2^k},\frac{\zeta}{2^k})$ where

$$m^{\#}(\xi,\zeta)=\frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\widetilde{\psi}_k(\xi+\zeta)\psi(\xi)\psi_2(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}$$

and $\operatorname{supp} m^{\#}\subset\{(\xi,\zeta)\in\mathbb{R}^N\times\mathbb{R}^N/\ \frac12<|\xi|<2,\ 2<|\zeta|<8,\ \frac1{16}<|\xi+\zeta|<16\}$.

Let $\widetilde{h},\widetilde{\eta}\in\mathcal{S}$ be such that $\widetilde{h}\equiv1$ on $\operatorname{supp}\psi$ with $\operatorname{supp}\widetilde{h}\subset\{\xi\in\mathbb{R}^N/\ \frac13<|\xi|<\frac73\}$ and $\widetilde{\eta}\equiv1$ on $\operatorname{supp}\psi_2$ with $\operatorname{supp}\widetilde{\eta}\subset\{\zeta\in\mathbb{R}^N/\ \frac13 2^2<|\zeta|<\frac73 2^2\}$. Then

$$m^{\#}(\xi,\zeta)=\frac{i(\zeta|\xi|^{2s}+\xi|\zeta|^{2s})\,\widetilde{\psi}(\xi+\zeta)\psi(\xi)\psi_2(\zeta)h(\xi)\eta(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}. \tag{2.21}$$

Moreover,

$$\begin{aligned}\frac{i(\xi|\zeta|^{2s}+\zeta|\xi|^{2s})\widetilde{h}(\xi)\widetilde{\eta}(\zeta)}{|\xi|^{2s}|\zeta|^{2s}}&=\frac{i\xi|\xi+\zeta|^2}{|\xi|^{2s}|\zeta|^2}\,\frac{|\zeta|^2\widetilde{h}(\xi)\widetilde{\eta}(\zeta)}{|\xi+\zeta|^2}+\frac{i\zeta|\xi+\zeta|^2}{|\zeta|^{2s}|\xi|^2}\,\frac{|\xi|^2\widetilde{h}(\xi)\widetilde{\eta}(\zeta)}{|\xi+\zeta|^2}\\ &:=\frac{i\xi|\xi+\zeta|^2}{|\xi|^{2s}|\zeta|^2}\tau_{22}^1(\xi,\zeta)+\frac{i\zeta|\xi+\zeta|^2}{|\zeta|^{2s}|\xi|^2}\tau_{22}^2(\xi,\zeta).\end{aligned}\tag{2.22}$$

Observe that $|\xi+\zeta| \sim |\xi| \sim |\zeta|$ for $(\xi,\zeta) \in \operatorname{supp}\widetilde{h} \times \operatorname{supp}\widetilde{\eta}$. Thus $\tau_{22}^1, \tau_{22}^2 \in \mathcal{S}(\mathbb{R}^N \times \mathbb{R}^N)$ and

$$\tau_{22}^j(\xi,\zeta) = c_N \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} e^{i(\mu\cdot\xi+\nu\cdot\zeta)} r_{22}^j(\mu,\nu)d\mu d\nu \quad \text{with } j=1,2. \tag{2.23}$$

It is easy to check that $r_{22}^1, \tau_{22}^2 \in \mathcal{S}(\mathbb{R}^N \times \mathbb{R}^N)$. Plugging (2.22)–(2.23) into (2.21) yields

$$\begin{aligned} m^{\#}(\xi,\zeta) = {} & c_N \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |\xi+\zeta|^2\widetilde{\psi}(\xi+\zeta)\, \frac{i\xi e^{i\mu\cdot\xi}\psi(\xi)}{|\xi|^{2s}} \frac{e^{i\nu\cdot\zeta}\psi_2(\zeta)}{|\zeta|^2} r_{22}^1(\mu,\nu)d\mu d\nu \\ & + c_N \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} |\xi+\zeta|^2\widetilde{\psi}(\xi+\zeta)\, \frac{e^{i\mu\cdot\xi}\psi(\xi)}{|\xi|^2} \frac{i\zeta e^{i\nu\cdot\zeta}\psi_2(\zeta)}{|\zeta|^{2s}} r_{22}^2(\mu,\nu)d\mu d\nu. \end{aligned} \tag{2.24}$$

By (2.15), (2.16), (2.19)–(2.21), (2.23), (2.24), $m_k^*(\xi,\zeta) = 2^{(1-2s)k}m^*(\frac{\xi}{2^k},\frac{\zeta}{2^k})$ and $m_k^{\#}(\xi,\zeta) = 2^{(1-2s)k}m^{\#}(\frac{\xi}{2^k},\frac{\zeta}{2^k})$ as well as (2.14), we can rewrite $\mathcal{H}_2(\rho,\rho)$ as follows

$$\begin{aligned} \mathcal{H}_2(\rho,\rho) = & -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \Delta P_{\le k-4}\big(\nabla K_{2+2s} * \Delta_k^\mu\rho\, \Delta_{k+2}^\nu\rho\big)\Big] r_{21}^1(\mu,\nu)d\mu d\nu \\ & -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \Delta\widetilde{\Delta}_k\big(\nabla K_{2s} * \Delta_k^\mu\rho\, K_2 * \Delta_{k+2}^\nu\rho\big)\Big] r_{22}^1(\mu,\nu)d\mu d\nu \\ & -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \Delta\widetilde{\Delta}_k\big(K_2 * \Delta_k^\mu\rho\, \nabla K_{2s} * \Delta_{k+2}^\nu\rho\big)\Big] r_{22}^2(\mu,\nu)d\mu d\nu \\ & +\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \nabla\cdot P_{\le k-4}\big(\nabla\nabla K_{2+2s} * \Delta_k^\mu\rho\, \Delta_{k+2}^\nu\rho\big)\Big] r_{21}^2(\mu,\nu)d\mu d\nu \\ & +\sum_{k\in\mathbb{Z}} \nabla P_{\le k-4}(\Delta_k\rho\, K_{2s} * \Delta_{k+2}\rho), \end{aligned} \tag{2.25}$$

where $r_{21}^1(\mu,\nu), r_{21}^2(\mu,\nu), r_{22}^1(\mu,\nu), r_{22}^2(\mu,\nu) \in \mathcal{S}(\mathbb{R}^N \times \mathbb{R}^N)$.

Similarly, we have

$$\begin{aligned} \mathcal{H}_\ell(\rho,\rho) = & -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \Delta P_{\le k-4}\big(\nabla K_{2+2s} * \Delta_k^\mu\rho\, \Delta_{k+\ell}^\nu\rho\big)\Big] r_{\ell 1}^1(\mu,\nu)d\mu d\nu \\ & -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \Delta\widetilde{\Delta}_k\big(\nabla K_{2s} * \Delta_k^\mu\rho\, K_2 * \Delta_{k+\ell}^\nu\rho\big)\Big] r_{\ell 2}^1(\mu,\nu)d\mu d\nu \\ & -\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \Big[\sum_{k\in\mathbb{Z}} \Delta\widetilde{\Delta}_k\big(K_2 * \Delta_k^\mu\rho\, \nabla K_{2s} * \Delta_{k+\ell}^\nu\rho\big)\Big] r_{\ell 2}^2(\mu,\nu)d\mu d\nu \end{aligned}$$

$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big[\sum_{k\in\mathbb{Z}}\nabla\cdot P_{\le k-4}\big(\nabla\nabla K_{2+2s}*\Delta_k^{\mu}\rho\,\Delta_{k+\ell}^{\nu}\rho\big)\Big]r_{\ell 1}^{2}(\mu,\nu)d\mu d\nu$$

$$+\sum_{k\in\mathbb{Z}}\nabla P_{\le k-4}(\Delta_k\rho\, K_{2s}*\Delta_{k+\ell}\rho), \tag{2.26}$$

where $r_{\ell 1}^1(\mu,\nu), r_{\ell 1}^2(\mu,\nu), r_{\ell 2}^1(\mu,\nu), r_{\ell 2}^2(\mu,\nu)\in\mathcal{S}(\mathbb{R}^N\times\mathbb{R}^N)$ and $\ell=0,1$.

Combining (2.12), (2.25) and (2.26), we complete the whole proof. □

Remark 2.2. Recall that $\Delta_k^{\mu}f=\mathcal{F}^{-1}\big(e^{i\mu\cdot 2^{-k}\xi}\psi(2^{-k}\xi)\hat{f}(\xi)\big)$. Then we have

$$\big(\Delta_k^{\mu}f\big)(x)=\big(\Delta_k f\big)(x+2^{-k}\mu). \tag{2.27}$$

Similarly, for any $\ell\in\mathbb{N}\cap[-2,2]$, from $\Delta_{k-\ell}^{\nu}g=\mathcal{F}^{-1}\big(e^{i\nu\cdot 2^{-k}\zeta}\psi(2^{-k+\ell}\zeta)\hat{g}(\zeta)\big)$, we have

$$\big(\Delta_{k-\ell}^{\nu}g\big)(x)=\big(\Delta_{k-\ell}f\big)(x+2^{-k}\nu). \tag{2.28}$$

It is clear that in the above proof, we used both the symmetric and the cancellation properties of $\rho\nabla K_{2s}*\rho$. Similarly, it is easy to check from (2.17) that the above decomposition also works for $f\nabla K_{2s}*g+g\nabla K_{2s}*f$ with $0\le s\le\frac{N}{2}$.

2.2. Smoothing effect and product estimates

In this subsection, we recall the smoothing effect of the heat equation:

$$\begin{cases}\rho_t-\mu_{en}\Delta\rho=F,\\ \rho(x,0)=0.\end{cases} \tag{2.29}$$

Lemma 2.3. *Let $(\sigma,q,r)\in(-\infty,\infty)\times[1,\infty]\times[1,2]$ and $F(x,t)\in\widetilde{L}_t^r(\dot{B}_{\infty,q}^{\sigma-2+\frac{2}{r}})$. Then the mild solution $\rho=\int_0^t e^{\mu_{en}(t-\tau)\Delta}F(x,\tau)d\tau$ to system (2.29) satisfies*

$$\|\rho\|_{\widetilde{L}_t^\infty(\dot{B}_{\infty,q}^{\sigma})\cap\widetilde{L}_t^2(\dot{B}_{\infty,q}^{\sigma+1})}\lesssim\|F\|_{\widetilde{L}_t^r(\dot{B}_{\infty,q}^{\sigma-2+\frac{2}{r}})}.$$

Proof. The proof is similar to [7, Lemma 2.1]. Hence we omit the details. □

Let $F=\mu_{in}\nabla\cdot(\rho\nabla K_{2s}*\rho)$ and $(\sigma,r)=(-2s,1)$ or $(\sigma,r)=(-2s,2)$. From Lemma 2.3 we have

$$\|\rho\|_{\widetilde{L}_t^\infty(\dot{B}_{\infty,q}^{-2s})\cap\widetilde{L}_t^2(\dot{B}_{\infty,q}^{-2s+1})}\lesssim\|F\|_{\widetilde{L}_t^2(\dot{B}_{\infty,q}^{-1-2s})+\widetilde{L}_t^1(\dot{B}_{\infty,q}^{-2s})}$$

where Γ is the set of all decompositions of F, i.e. $\Gamma=\{(F_1,F_2)/\ F=F_1+F_2\}$, and

$$\|F\|_{\widetilde{L}_t^2(\dot{B}_{\infty,q}^{-1-2s})+\widetilde{L}_t^1(\dot{B}_{\infty,q}^{-2s})}=\min_{(F_1,F_2)\in\Gamma}\Big(\|F_1\|_{\widetilde{L}_t^2(\dot{B}_{\infty,q}^{-1-2s})}+\|F_2\|_{\widetilde{L}_t^1(\dot{B}_{\infty,q}^{-2s})}\Big).$$

The next lemma is a key application of Lemma 2.1. In particular, based on the cancellation property of $\Pi_l^{hh}(\rho, \nabla K_{2s} * \rho) + \Pi_h^{hh}(\rho, \nabla K_{2s} * \rho)$, we observe from (2.8) that $\rho\nabla K_{2s} * \rho$ can be formally thought as the summation of first order derivatives of several controllable quadratic terms of ρ, see e.g. (2.8).

Noticing that the additional one order derivative ∇ ensures that $2-2s > 0$ $(0 \le s < 1)$. Hence we can prove the existence/uniqueness of the solution to (IGD).

Lemma 2.4. *Let* $(s, q) \in [0, 1) \times [1, \infty]$. *Then we have*

$$\|\rho\nabla K_{2s} * \rho\|_{\widetilde{L}_t^1(\dot{B}_{\infty,q}^{1-2s})+\widetilde{L}_t^2(\dot{B}_{\infty,q}^{-2s})} \lesssim \|\rho\|^2_{\widetilde{L}_t^2(\dot{B}_{\infty,q}^{1-2s})\cap\widetilde{L}_t^\infty(\dot{B}_{\infty,q}^{-2s})}. \tag{2.30}$$

Proof. Recall from (2.8) that

$$\rho\nabla K_{2s} * \rho = \Delta T_{s,1}(\rho,\rho) + \nabla T_{s,2}(\rho,\rho) + \nabla\cdot T_{s,3}(\rho,\rho).$$

In order to prove (2.30), we shall apply a case by case arguments. It suffices to estimate

$$\|\rho\nabla K_{2s} * \rho\|_{\widetilde{L}_t^1(\dot{B}_{\infty,q}^{1-2s})+\widetilde{L}_t^2(\dot{B}_{\infty,q}^{-2s})} \lesssim \|T_{s,1}\|_{\widetilde{L}_t^2(\dot{B}_{\infty,q}^{2-2s})} + \|\nabla T_{s,2} + \nabla\cdot T_{s,3}\|_{\widetilde{L}_t^1(\dot{B}_{\infty,q}^{1-2s})}.$$

Let us do some preliminary calculation. Recall that L_x^∞ is a shift-invariant Banach space. Then for any $\mu \in \mathbb{R}^N$, from (2.27)–(2.28) we have

$$\sup_{\mu\in\mathbb{R}^N} \|\nabla K_{2+2s} * \Delta_k^\mu \rho\|_{L_x^\infty} \lesssim 2^{(-1-2s)k}\|\Delta_k f\|_{L_x^\infty}. \tag{2.31}$$

Indeed,

$$\begin{aligned}\nabla K_{2+2s} * \Delta_k^\mu \rho &= \mathcal{F}^{-1}\big(i\xi|\xi|^{-2-2s}\widetilde{\psi}(2^{-k}\xi)e^{i\mu\cdot 2^{-k}\xi}\psi(2^{-k}\xi)\widehat{f}(\xi)\big)\\ &:= \nabla K_{2+2s} * \widetilde{\Delta}_k^\mu \Delta_k \rho.\end{aligned}$$

By direct calculation, we get

$$\big|\mathcal{F}^{-1}\big(i\xi\,|\xi|^{-2-2s}e^{i\mu\cdot 2^{-k}\xi}\widetilde{\psi}(2^{-k}\xi)\big)(x)\big| \lesssim 2^{-(1+2s)k+kN}\big(1+|2^k x+\mu|\big)^{-N-1}.$$

From Young's inequality and the fact that L_x^1 is also a shift-invariant Banach space we get

$$\begin{aligned}\sup_{\mu\in\mathbb{R}^N} \|\nabla K_{2s} * \Delta_k^\mu \rho\|_{L_x^\infty} &\le c_{s,N}\,\|\Delta_k f\|_{L_x^\infty} \sup_\mu \int_{\mathbb{R}^N} \frac{2^{-(1+2s)k+kN}}{(1+|2^k x+\mu|)^{N+1}}dx\\ &\le c_{s,N}\, 2^{-(1+2s)k}\|\Delta_k f\|_{L_x^\infty}.\end{aligned} \tag{2.32}$$

Similarly, we have

$$\sup_{\nu\in\mathbb{R}^N} \|\Delta_{k-\ell}^\nu g\|_{L_x^\infty} \lesssim \|\Delta_{k-\ell} g\|_{L_x^\infty}. \tag{2.33}$$

Estimate of $\|T_{s,1}\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})}$. It suffices to estimate

$$\big\|K_2 * \Pi^{hl}_h(\rho, \nabla K_{2s} * \rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} + \big\|K_2 * \Pi^{lh}_h(\rho, \nabla K_{2s} * \rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})}$$

and

$$\begin{cases} \sup_{\mu,\nu} \big\|\sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} P_{\le k-4}(\nabla K_{2+2s} * \Delta^\mu_k \rho\, \Delta^\nu_{k+\ell}\rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})}, \\ \sup_{\mu,\nu} \big\|\sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} \widetilde{\Delta}_k(\nabla K_{2s} * \Delta^\mu_k \rho\, K_2 * \Delta^\nu_{k+\ell}\rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})}, \\ \sup_{\mu,\nu} \big\|\sum_{k\in\mathbb{Z}} \sum_{\ell=0,1,2} \widetilde{\Delta}_k(K_2 * \Delta^\mu_k \rho\, \nabla K_{2s} * \Delta^\nu_{k+\ell}\rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} \end{cases}$$

since $\|r^j_{il}\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} \lesssim 1$ for any $(i, j, l) \in \{0, 1, 2\} \times \{1, 2\} \times \{1, 2\}$.

Recall that $\Pi^{hl}_h(\rho, \nabla K_{2s} * \rho) = \sum_{k\in\mathbb{Z}} \widetilde{\Delta}_k(\Delta_k\rho\, P_{\le k-3}\nabla K_{2s} * \rho)$. Then it is easy to check that the Fourier transform of $\Delta_k\rho\, P_{\le k-3}\nabla K_{2s} * \rho$ has a compact support near $|\xi| \sim 2^k$. Therefore, from Definition 1.6 we get

$$\begin{aligned} \big\|K_2 * \Pi^{hl}_h(\rho, \nabla K_{2s} * \rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} &\lesssim \big\|\Pi^{hl}_h(\rho, \nabla K_{2s} * \rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{-2s}_{\infty,q})} \\ &\lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{-2skq}\|\Delta_k\rho\, \nabla K_{2s} P_{\le k-3}\rho\|^q_{L^2_t L^\infty_x}\Big)^{\frac1q} \\ &\lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{-2skq}\|\Delta_k\rho\|^q_{L^2_t L^\infty_x}\|\nabla K_{2s} * P_{\le k-3}\rho\|^q_{L^\infty_t L^\infty_x}\Big)^{\frac1q}. \end{aligned} \tag{2.34}$$

Recall that $\ell^q \subset \ell^\infty$, i.e. $\sup_k |a_k| \lesssim \big(\sum_k |a_k|^q\big)^{\frac1q}$ for any $\{a_k\}_{k\in\mathbb{Z}} \in \ell^q$ and $1 \le q \le \infty$. Then we get

$$\begin{aligned} \sup_{k\in\mathbb{Z}} 2^{-k}\|\nabla K_{2s} * P_{\le k-3}\rho\|_{L^\infty_t L^\infty_x} &\lesssim \sup_{k\in\mathbb{Z}} 2^{-k}\Big\|\sum_{j\le k-3} \nabla K_{2s} * \Delta_j\rho\Big\|_{L^\infty_t L^\infty_x} \\ &\lesssim \sup_{k\in\mathbb{Z}} \sum_{j\le k-3} 2^{j-k}2^{-2sj}\big\|\Delta_j\rho\big\|_{L^\infty_t L^\infty_x} \lesssim \|g\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})} \end{aligned} \tag{2.35}$$

where in the second inequality we used Minkowski's inequality and Bernstein's inequality, and in the last inequality we used Young's inequality, i.e. $\ell^{\frac{q}{q-1}} * \ell^q \to \ell^\infty$ and $\sum_{j\le -3} 2^{-j} \le 1$.

Plugging (2.35) into (2.34), we get

$$\big\|K_2 * \Pi^{hl}_h(\rho,\, \nabla K_{2s} * \rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} \lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})}. \tag{2.36}$$

Similarly,

$$\begin{aligned} \big\|K_2 * \Pi^{lh}_h(\rho,\nabla K_{2s} * \rho)\big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} &\lesssim \big\|\{2^{(1-4s)k}\|P_{\le k-3}\rho\|_{L^\infty_t L^\infty_x}\|\Delta_k\rho\|_{L^2_t L^\infty_x}\}_k\big\|_{\ell^q} \\ &\lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})} \sup_{k\in\mathbb{Z}} 2^{-2sk}\big\|P_{\le k-3}\rho\big\|_{L^\infty_t L^\infty_x} \end{aligned}$$

$$\lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})}. \tag{2.37}$$

Applying (2.32) and (2.34) to $\nabla K_{2+2s} * \Delta^\mu_k\rho\,\Delta^\nu_{k+\ell}\rho$ yields

$$\begin{aligned}
&\sup_{\mu,\nu}\Big\|\sum_{k\in\mathbb{Z}}\sum_{\ell=0,1,2} P_{\le k-4}(\nabla K_{2+2s} * \Delta^\mu_k\rho\,\Delta^\nu_{k+\ell}\rho)\Big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})}\\
&= \sup_{\mu,\nu}\Big(\sum_{j\in\mathbb{Z}} 2^{(2-2s)jq}\Big\|\sum_{k\ge j+1}\sum_{\ell=0,1,2}\Delta_j P_{\le k-4}(\nabla K_{2+2s} * \Delta^\mu_k\rho\,\Delta^\nu_{k+\ell}\rho)\Big\|^q_{L^2_tL^\infty_x}\Big)^{\frac1q}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\ge j+1}\sum_{\ell=0,1,2} 2^{(2-2s)(j-k)+(1-4s)k}\|\Delta_k\rho\|_{L^2_tL^\infty_x}\|\Delta_{k+\ell}\rho\|_{L^\infty_tL^\infty_x}\Big)^q\Big)^{\frac1q}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}}\sum_{\ell=0,1,2} 2^{(1-4s)kq}\|\Delta_k\rho\|^q_{L^2_tL^\infty_x}\|\Delta_{k+\ell}\rho\|^q_{L^\infty_tL^\infty_x}\Big)^{\frac1q}\\
&\lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})}.
\end{aligned}\tag{2.38}$$

Similarly, for any $\mu,\nu\in\mathbb{R}^N$ and $\ell=0,1,2$ we get

$$\Big\|\sum_{k\in\mathbb{Z}}\widetilde{\Delta}_k(\nabla K_{2s} * \Delta^\mu_k\rho\,K_2 * \Delta^\nu_{k+\ell}\rho)\Big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} \lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})}$$

$$\Big\|\sum_{k\in\mathbb{Z}}\widetilde{\Delta}_k(K_2 * \Delta^\mu_k\rho)\,\nabla K_{2s} * \Delta^\nu_{k+\ell}\rho\Big\|_{\widetilde{L}^2_t(\dot{B}^{2-2s}_{\infty,q})} \lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})}.$$

Combining (2.36)–(2.38) and the above two estimates we get the desired estimates for $T_{s,1}(\rho,\rho)$.

Estimate of $\|T_{s,2}(\rho,\rho)\|_{\widetilde{L}^1_t(\dot{B}^{2-2s}_{\infty,q})} + \|T_{s,3}(\rho,\rho)\|_{\widetilde{L}^1_t(\dot{B}^{2-2s}_{\infty,q})}$. By using (2.32) and (2.34), we observe that $T_{s,2}(\rho,\rho)$ and $T_{s,3}(\rho,\rho)$ can be treated in the similar way. As a consequence, it suffices to estimate $\|T_{s,3}(\rho,\rho)\|_{\widetilde{L}^1_t(\dot{B}^{2-2s}_{\infty,q})}$. Similar to (2.38), we get

$$\begin{aligned}
&\sup_{\mu,\nu}\Big\|\sum_{k\in\mathbb{Z}}\sum_{\ell=0}^{2} P_{\le k-4}(\nabla\nabla K_{2+2s} * \Delta^\mu_k\rho\,\Delta^\nu_{k+\ell}\rho)\Big\|_{\widetilde{L}^1_t(\dot{B}^{2-2s}_{\infty,q})}\\
&= \sup_{\mu,\nu}\Big(\sum_{j\in\mathbb{Z}} 2^{(2-2s)jq}\Big\|\sum_{k\ge j+1}\sum_{\ell=0}^{2}\Delta_j P_{\le k-4}(\nabla\nabla K_{2+2s} * \Delta^\mu_k\rho\,\Delta^\nu_{k+\ell}\rho)\Big\|^q_{L^1_tL^\infty_x}\Big)^{\frac1q}\\
&\lesssim \Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\ge j+1}\sum_{\ell=0}^{2} 2^{(2-2s)(j-k)+(2-4s)k}\|\Delta_k\rho\|_{L^2_tL^\infty_x}\|\Delta_{k+\ell}\rho\|_{L^2_tL^\infty_x}\Big)^q\Big)^{\frac1q}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}}\sum_{\ell=0}^{2} 2^{(2-4s)kq}\|\Delta_k\rho\|^q_{L^2_tL^\infty_x}\|\Delta_{k+\ell}\rho\|^q_{L^2_tL^\infty_x}\Big)^{\frac1q}\\
&\lesssim \|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,2q})}\|\rho\|_{\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,2q})}.
\end{aligned}$$

Combining the above estimates for $T_{s,1}(\rho,\rho)$, $T_{s,2}(\rho,\rho)$ and $T_{s,3}(\rho,\rho)$, we complete the proof of (2.30). □

2.3. Characterization of homogeneous Besov space

The characterizations of homogeneous Besov space $\dot{B}^{-2s}_{\infty,q}$ are as follows.

Lemma 2.5. *Let $(s, \mu_{en}, q, p) \in [0, 1) \times (0, \infty) \times [1, \infty]^2$. Then we get*

$$f \in \dot{B}^{-2s}_{\infty,q} \Longleftrightarrow e^{\mu_{en} t\Delta} f \in \widetilde{L}^p_t(\dot{B}^{-2s+\frac{2}{p}}_{\infty,q}).$$

Moreover, for any $(s, \mu_{en}) \in (0, 1) \times (0, \infty)$, we have

$$f \in \dot{B}^{-2s}_{\infty,\infty} \Longleftrightarrow \sup_{r>0, x\in\mathbb{R}^N} \frac{1}{r^{N+2-2s}} \int_{Q(x,t;r)} |e^{\mu_{en} t\Delta} \Lambda^{-s} f|^2 dydt < \infty.$$

In particular, the above results still work when f is replaced by[5] $\frac{\nabla}{\Lambda} f$.

Proof. Part 1 If $e^{\mu_{en} t\Delta} f \in \widetilde{L}^p_t(\dot{B}^{-2s+\frac{2}{p}}_{\infty,q})$, then from Definition 1.6 we have

$$\Big(\sum_{k\in\mathbb{Z}} 2^{k(-2s+\frac{2}{p})q} \|e^{\mu_{en} t\Delta} \Delta_k f\|^q_{L^p_t L^\infty_x}\Big)^{\frac{1}{q}} < \infty.$$

By direct computation we get

$$\Delta_k f = \frac{2^{2k}}{3} \int_{2^{-2k}}^{2^{2-2k}} e^{-\mu_{en} t\Delta}\, e^{\mu_{en} t\Delta} \Delta_k f dt.$$

As a consequence of Bernstein's inequality and Minkowski's inequality we get

$$\big\|\Delta_k f\big\|_{L^\infty_x} \lesssim 2^{2k} \int_{2^{-2k}}^{2^{2-2k}} \|\Delta_k e^{\mu_{en} t\Delta} f\|_{L^\infty_x} dt \lesssim 2^{\frac{2k}{p}} \|\Delta_k e^{\mu_{en} t\Delta} f\|_{L^p_t L^\infty_x}.$$

Then summing up $2^{-2sk}\|\Delta_k f\|_{L^\infty_x}$ and using Definition 1.6 yields

$$\|f\|_{\dot{B}^{-2s}_{\infty,q}} \lesssim \|e^{\mu_{en} t\Delta} f\|_{\widetilde{L}^p_t(\dot{B}^{-2s+\frac{2}{p}}_{\infty,q})}. \tag{2.39}$$

Since $2^{\frac{2k}{p}} e^{-c\mu_{en} t 2^{2k}}$ is uniformly bounded in L^p_t, it is easy to check that

$$2^{k(\frac{2}{p}-2s)} \|e^{\mu_{en} t\Delta} \Delta_k f\|_{L^p_t L^\infty_x} \lesssim \frac{\|\Delta_k f\|_{L^\infty_x}}{2^{2ks}} \big\|2^{\frac{2k}{p}} e^{-c\mu_{en} 2^{2k}\cdot}\big\|_{L^p_t} \lesssim \frac{\|\Delta_k f\|_{L^\infty_x}}{2^{2ks}}.$$

[5] Riesz transforms are bounded operators in homogeneous Besov spaces.

Hence applying Definition 1.6 to the above inequality gives

$$\|e^{\mu_{en}t\Delta}f\|_{\widetilde{L}^p_t(\dot{B}^{-2s+\frac{2}{p}}_{\infty,q})} \lesssim \|f\|_{\dot{B}^{-2s}_{\infty,q}}. \tag{2.40}$$

Combining (2.39) and (2.40), we prove the first result of this Lemma.

Part 2 For any $0 < s < 1$ and $\mu_{en} > 0$, in order to prove

$$f \in \dot{B}^{-2s}_{\infty,\infty} \Leftrightarrow \Lambda^{-s}f \in \dot{B}^{-s}_{\infty,\infty} \Leftrightarrow \sup_{r>0,x} \frac{1}{r^{N+2-2s}} \int\limits_{Q(x,t;r)} |e^{\mu_{en}t\Delta}\Lambda^{-s}f|^2 dydt < \infty,$$

using $g \in \dot{B}^{-s}_{\infty,\infty} \Leftrightarrow \sup_{t>0} t^{\frac{s}{2}}\|e^{\mu_{en}t\Delta}g\|_{L^\infty_x} < \infty$ (cf. [21]), it suffices to show

$$\sup_{t>0} t^{\frac{s}{2}}\|e^{\mu_{en}t\Delta}g\|_{L^\infty_x} < \infty \Leftrightarrow \sup_{r>0,x} \frac{1}{r^{N+2-2s}} \int\limits_{Q(x,t;r)} |e^{\mu_{en}t\Delta}g|^2 dydt < \infty$$

On the one hand, it is quite straightforward that

$$\begin{aligned} \int\limits_{Q(x,t;r)} |e^{\mu_{en}t\Delta}g|^2 dydt &\lesssim \sup_{t>0} t^s \|e^{\mu_{en}t\Delta}g\|^2_{L^\infty_x} \int\limits_0^{r^2}\int\limits_{B(x;r)} \frac{dydt}{t^s} \\ &\lesssim r^{N+2-2s}(\sup_{t>0} t^{\frac{s}{2}}\|e^{\mu_{en}t\Delta}g\|_{L^\infty_x})^2. \end{aligned}$$

On the other hand, it is easy to get

$$e^{\mu_{en}t\Delta}g(x) = \frac{2}{t}\int\limits_0^{\frac{t}{2}}\int\limits_{\mathbb{R}^N} \frac{(2\pi\mu_{en})^{-\frac{N}{2}}}{(t-s)^{\frac{N}{2}}} e^{-\frac{|y|^2}{4\mu_{en}(t-s)}} (e^{\mu_{en}s\Delta}g)(x-y)dyds.$$

As a consequence, we obtain that

$$\begin{aligned} t^{\frac{s}{2}}|e^{\mu_{en}t\Delta}g(x)| &\lesssim \sum_{k\in\mathbb{Z}^N}^{\infty} e^{-|k|^2} \frac{1}{t^{\frac{N+2-s}{2}}} \int\limits_0^t \int\limits_{\frac{y}{\sqrt{4\mu_{en}t}}\in k+[0,1]^N} |e^{\mu_{en}s\Delta}g(x-y)|dyds \\ &\lesssim \sup_{k\in\mathbb{Z}^N}\Big(\frac{1}{t^{\frac{N+2-2s}{2}}} \int\limits_0^t \int\limits_{\frac{y}{\sqrt{4\mu_{en}t}}\in k+[0,1]^N} |e^{\mu_{en}s\Delta}g(x-y)|^2 dyds\Big)^{\frac{1}{2}} \\ &\lesssim \Big(\sup_{r>0,z\in\mathbb{R}^N} \frac{1}{r^{N+2-2s}} \int\limits_0^{r^2}\int\limits_{B(z;r)} |e^{\mu_{en}s\Delta}g(y)|^2 dyds\Big)^{\frac{1}{2}} \end{aligned}$$

which concludes the desired estimate of the second result of this Lemma and finishes the whole proof. □

2.4. Continuity of the heat semigroup in various spaces

Recall that $e^{t\Delta}$ is a strongly continuous semigroup in L^p_x ($p \in [1,\infty)$) and various other spaces. However, it is known that Schwartz space is not dense in $L^\infty_x \subset BMO$, hence $e^{t\Delta}$ is not a continuous semigroup in L^∞_x and BMO. Meanwhile, in the bounded uniform continuous function space (a subspace of L^∞_x), Giga proved that the heat semigroup $e^{t\Delta}$ is a continuous semigroup in BUC (cf. [10]). It is easy to check that $\dot{B}^0_{\infty,1} \subset BUC$ in which $e^{t\Delta}$ also generates a continuous semigroup. Furthermore, we can extend the proof to homogeneous Besov spaces $\dot{B}^{-2s}_{\infty,q}$ with $0 \le s \le 1$ and $1 \le q < \infty$.

Definition 2.6. A family of bounded operators $\{T(t),\ 0 \le t \le \infty\}$ on a Banach space X is called a strongly continuous semigroup if:
(1) $T(0) = I_d$,
(2) $T(t_1)T(t_2) = T(t_1+t_2)$, $\forall\, t_1, t_2 > 0$,
(3) for any $x \in X$, $x \mapsto T(t)x$ is continuous.

Proposition 2.7. *For any* $(s,q) \in [0,1] \times [1,\infty)$, $e^{t\Delta}$ *is a strongly continuous semigroup in* $\dot{B}^{-2s}_{\infty,q}$.

Proof. It suffices to prove that for any $f \in \dot{B}^{-2s}_{\infty,q}$,

$$\lim_{t\downarrow 0} \|e^{t\Delta} f - f\|_{\dot{B}^{-2s}_{\infty,q}} = 0. \tag{2.41}$$

Indeed, for given $f \in \dot{B}^{-2s}_{\infty,q}$, we have $c_f := (\sum 2^{-2sqk} \|\Delta_k f\|^q_{L^\infty_x})^{\frac{1}{q}} < \infty$. Then for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\Big(\sum_{k \ge N_\varepsilon} 2^{-2sqk} \|\Delta_k e^{t\Delta} f\|^q_{L^\infty_x}\Big)^{\frac{1}{q}} \le \Big(\sum_{k \ge N_\varepsilon} 2^{-2sqk} \|\Delta_k f\|^q_{L^\infty_x}\Big)^{\frac{1}{q}} < \frac{\varepsilon}{4}. \tag{2.42}$$

Meanwhile, fix N_ε, for any $0 < t < \frac{\varepsilon}{2^{2N_\varepsilon+6} c_f}$,

$$\begin{aligned}\Big(\sum_{k=-\infty}^{N_\varepsilon-1} 2^{-2skq} \|e^{t\Delta}\Delta_k f - \Delta_k f\|^q_{L^\infty_x}\Big)^{\frac{1}{q}} &\le \Big(\sum_{k=-\infty}^{N_\varepsilon-1} 2^{-2skq} (1-e^{-t2^{2N+6}})^q \|\Delta_k f\|^q_{L^\infty_x}\Big)^{\frac{1}{q}} \\ &\le t2^{2N+6}\Big(\sum_{k=-\infty}^{N_\varepsilon-1} 2^{-2skq} \|\Delta_k f\|^q_{L^\infty_x}\Big)^{\frac{1}{q}} \le c_f 2^{2N_\varepsilon+6} t < \frac{\varepsilon}{2}. \end{aligned}\tag{2.43}$$

Combining (2.42) and (2.43) yields (2.41). □

3. Proof of the main results

In this section, we shall give a case by case analysis of the global well-posedness of the following general diffusion system:

$$\rho_t - \mu_{en}\Delta\rho - \mu_{in}\nabla\cdot(\rho\nabla\Lambda^{-2s}\rho) = 0 \tag{3.1}$$

with initial data ρ_0 and $0 \le s \le 1$ and $(s, N) = (2, 4)$.

3.1. Analysis of (3.1) with $s = 0$

In this subsection, when $s = 0$, we get from (3.1) with initial data ρ_0 that

$$\rho_t - \mu_{en}\Delta\rho - \frac{\mu_{in}}{2}\Delta(\rho^2) = 0, \quad \rho|_{t=0} = \rho_0, \tag{$GD_{s=0}$}$$

where, in general, ρ is assumed to be nonnegative.

As is stated in the introduction, scaling invariant suggests that the right space should be L^∞_x. Then one may ask whether $(GD_{s=0})$ admits a unique solution if ρ_0 is large in L^∞_x. Generally speaking, it is difficult to apply semigroup method to establish well-posedness of the large data Cauchy problem without using any a priori estimate. However, if the system has a priori energy estimate which, in addition, satisfies scaling invariant property, then it would be possible to combine the a priori estimate with local existence of mild with large data to achieve the goal.

Next we recall that $\int \rho(x,t)dx = \int \rho_0(x)dx$ and

$$\int \rho(x,t)\rho(x,t)dx \le C(N, \mu_{en}, \mu_{in}, \rho_0) \tag{3.2}$$

if $\mu_{in} > 0$ and $-\int_0^t \int \rho|u|^2 dx \le 0$ since from (1.6) we have

$$\int \mu_{en}\rho(x,t)\ln\rho(x,t) + \frac{\mu_{in}}{2}(\rho(x,t))^2 dx \le \int \mu_{en}\rho_0 \ln\rho_0 + \frac{\mu_{in}}{2}\rho_0^2 dx.$$

It seems impossible to apply to a priori estimates (3.2) to L^∞_x solution. However, it is still possible to investigate the small perturbation of $(GD_{s=0})$ near large positive constant state, which can be thought as a special large data solution with respect to the original problem. For example, let $\rho = 1 + \tilde{\rho}$. Then we get

$$\tilde{\rho}_t - (\mu_{en} + \mu_{in})\Delta\tilde{\rho} = \frac{\mu_{in}}{2}\Delta(\tilde{\rho}^2).$$

It is clear that $\mu_{en} + \mu_{in}$ can be positive, zero and negative, which affects the essential structures, i.e. $\tilde{\rho}_t - (\mu_{en} + \mu_{in})\Delta\tilde{\rho}$.

Conclusively, if $\mu_{in} \ge 0$, then we can linearize system $(GD_{s=0})$ near any nonnegative constant state and establish the existence of mild solution (small perturbation); else if $\mu_{in} < 0$, then sufficiently small perturbation near any positive constant state less than $\frac{\mu_{en}}{-\mu_{in}}$ also works; else if $\mu_{in} < 0$ and the positive constant is bigger than $-\frac{\mu_{en}}{\mu_{in}}$, then we might have finite time blow up similar to the Keller–Segel system.

Usually, one can deal with the small perturbation near large positive constant state problem by using the similar way of the corresponding small data Cauchy problem. Thus we consider small initial data problem below. Let

$$\mathcal{J}_0:\ \rho \mapsto \mathcal{J}_0(\rho) = e^{\mu_{en} t\Delta}\rho_0 + \frac{\mu_{in}}{2}\int_0^t e^{\mu_{en}(t-\tau)\Delta}\Delta(\rho^2)d\tau. \tag{3.3}$$

Next we will prove the a priori estimate of $\mathcal{J}_0(\rho)$.

Proposition 3.1. *Let $\mathcal{J}_0$ be defined in (3.3). Assume that $(\mu_{en}, \mu_{in}) \in (0,\infty)^2$ and $\rho_0 \in \dot{B}^0_{\infty,1}$. Then we have*

$$\left\|\mathcal{J}_0(\rho)\right\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})^2\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})} \leq c\left\|\rho_0\right\|_{\dot{B}^0_{\infty,1}} + c_N\left\|\rho\right\|^2_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})}. \tag{3.4}$$

Additionally,

$$e^{\mu_{en}t\Delta}\rho_0 \to \rho_0 \ \ in\ \dot{B}^0_{\infty,1}\ \ as\ t\downarrow 0. \tag{3.5}$$

Proof. Applying Lemma 2.3 to (3.3) with $F = \frac{\mu_{in}}{2}\Delta(\rho^2)$, $(s,q,r) = (0,1,2)$ we have

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})} \leq \|e^{\mu_{en}t\Delta}\rho_0\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})} + c_N\|\rho^2\|_{\widetilde{L}^2_t(\dot{B}^1_{\infty,1})}.$$

Then applying Lemma 2.4 to the above estimate and applying Lemma 2.5 to $\|e^{\mu_{en}t\Delta}\rho_0\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})}$ yields (3.4). The time continuity of heat semigroup in $\dot{B}^0_{\infty,1}$, i.e. (3.5), follows from Proposition 2.7 with $(s,q) = (0,1)$. □

Proof of Theorem 1.7. We divide the proof into three steps. At first, Proposition 3.1 ensures that $\mathcal{J}_0$ maps a closed ball $\overline{B(0;\varepsilon)}$ of $\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1}) \cap \widetilde{L}^2_t(\dot{B}^1_{\infty,1})$ with $\varepsilon < 1/(4cc_N)$ into itself. Hence $\mathcal{J}_0$ is well defined. Next, suppose ρ_1 and ρ_2 are two solutions of (3.6) with the same initial data $\rho_0 \in \overline{B(0;\varepsilon)}$, then

$$\|\mathcal{J}_0(\rho_1) - \mathcal{J}_0(\rho_2)\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})} \leq 4cc_N\varepsilon\|\rho_1 - \rho_2\|_{\widetilde{L}^\infty_t(\dot{B}^0_{\infty,1})\cap\widetilde{L}^2_t(\dot{B}^1_{\infty,1})}$$

where $4cc_N\varepsilon < 1$. Thus existence and uniqueness of solutions follow immediately from contraction arguments. At last, time continuity follows from (3.5). Conclusively, we finish the proof of Theorem 1.7. □

3.2. Analysis of (3.1) with $0 < s < 1$

In this subsection, we study the following system

$$\rho_t - \mu_{en}\Delta\rho - \mu_{in}\nabla\cdot(\rho\nabla\Lambda^{-2s}\rho) = 0, \quad \rho|_{t=0} = \rho_0. \tag{$GD_{0<s<1}$}$$

Define

$$\mathcal{J}_s:\ \rho \mapsto \mathcal{J}_s(\rho) = e^{\mu_{en}t\Delta}\rho_0 + \mu_{in}\int_0^t e^{\mu_{en}(t-\tau)\Delta}\nabla\cdot(\rho\nabla\Lambda^{-2s}\rho)d\tau. \tag{3.6}$$

Next we will prove the a priori estimate of $\mathcal{J}_s(\rho)$.

Proposition 3.2. *Let $\mathcal{J}_s$ be as in (3.6). For any $(\mu_{en},\mu_{in},q) \in (0,\infty)^2 \times [1,\infty]$ and $\rho_0 \in \dot{B}^{-2s}_{\infty,q}$, we get*

$$\|\mathcal{J}_s(\rho)\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})} \le c\,\|\rho_0\|_{\dot{B}^{-2s}_{\infty,q}} + c_{N,s}\|\rho\|^2_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}. \tag{3.7}$$

Additionally, for any $1 \le q < \infty$,

$$e^{\mu_{en}t\Delta}\rho_0 \to \rho_0 \ \textit{in}\ \dot{B}^{-2s}_{\infty,q}\ \textit{as}\ t\downarrow 0.$$

Proof. Applying Lemma 2.3 to (3.6) with $F = \frac{\mu_{in}}{2}\nabla\cdot(\rho\nabla\Lambda^{-2s}\rho + \nabla\Lambda^{-2s}\rho)$ and $(s,q,r) \in (0,1)\times[1,\infty]\times\{1,2\}$ we have

$$\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})} \le \|e^{\mu_{en}t\Delta}\rho_0\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})} + \|\rho\nabla\Lambda^{-2s}\rho\|_X$$

where $X = \widetilde{L}^1_t(\dot{B}^{1-2s}_{\infty,q}) + \widetilde{L}^2_t(\dot{B}^{-2s}_{\infty,q})$. Then by Lemmas 2.4 and 2.5 we get (3.7). The time continuity of heat semigroup in $\dot{B}^{-2s}_{\infty,q}$ follows from Proposition 2.7 with $(s,q) \in (0,1)\times[1,\infty)$. □

Proof of Theorem 1.9. We divide the proof into three steps. At first, Proposition 3.1 ensures that $\mathcal{J}_s$ maps a closed ball $\overline{B(0;\varepsilon)}$ of $\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})$ with $\varepsilon < 1/(4cc_{N,s})$ into itself. Hence $\mathcal{J}_s$ is well defined. Next, suppose ρ_1 and ρ_2 are two solutions of (3.6) with the same initial data $\rho_0 \in \overline{B(0;\varepsilon)}$, then

$$\|\mathcal{J}_s(\rho_1)-\mathcal{J}_s(\rho_2)\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})} \le 4cc_{N,s}\varepsilon\|\rho_1-\rho_2\|_{\widetilde{L}^\infty_t(\dot{B}^{-2s}_{\infty,q})\cap\widetilde{L}^2_t(\dot{B}^{1-2s}_{\infty,q})}$$

where $4cc_{N,s}\varepsilon < 1$. Indeed, denote $\rho_j^{(2s)} = \Lambda^{-2s}\rho_j$. Then we have

$$2\,(\rho_1\nabla\rho_1^{(2s)} - \rho_2\nabla\rho_2^{(2s)}) = (\rho_1+\rho_2)\nabla(\rho_1-\rho_2)^{(2s)} + (\rho_1-\rho_2)\nabla(\rho_1+\rho_2)^{(2s)}.$$

It is clear that the right hand side of the above identity is symmetric and satisfies Lemmas 2.1 and 2.4. Thus existence and uniqueness of solutions follow immediately from contraction arguments. At last, time continuity follows from Proposition 3.2. Conclusively, we finish the proof of Theorem 1.9. □

3.3. Analysis of (3.1) with $s = 1$

In this subsection, we first recall the following bilinear estimates, see for instance [17, Lemmas 3.1 and 3.2, p. 28] and [21, Lemma 16.3, p. 163].

Lemma 3.3. *For any $N \in \mathbb{N} \cap [2, \infty)$, the bilinear operator $\mathcal{B}$ defined by*

$$\mathcal{B}(U, V) = \int_0^t e^{\mu_{en}(t-\tau)\Delta} \nabla \mathcal{R}_i \mathcal{R}_j \cdot (U \otimes V) d\tau \tag{3.8}$$

is continuous from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{E}$, where $\mathcal{E} \subset L^2_{uloc}$ and

$$U, V \in \mathcal{E} \Leftrightarrow \sup_{t>0} t^{\frac{1}{2}} \|U\|_{L^\infty_x} + \sup_{t>0} t^{\frac{1}{2}} \|V\|_{L^\infty_x} + \|U\|_{L^2_{uloc}} + \|V\|_{L^2_{uloc}} < \infty.$$

Remark 3.4. The above estimate also works when replacing $e^{\mu_{en}(t-\tau)\Delta} \nabla \mathcal{R}_i \mathcal{R}_j$ by $e^{\mu_{en}(t-\tau)\Delta} \Lambda$.

Recall that $\Delta\phi\nabla\phi = \nabla \cdot (\nabla\phi \otimes \nabla\phi) - \frac{\nabla}{2}\big(|\nabla\phi|^2\big)$. Then (3.1) is reduced to

$$\rho_t - \mu_{en}\Delta\rho + \mu_{in}\nabla \cdot (\rho \nabla \frac{1}{\Delta}\rho) = 0. \tag{3.9}$$

It is clear that if we denote $V = \nabla\frac{1}{\Delta}\rho$, then $\nabla \cdot (\rho \nabla \frac{1}{\Delta}\rho) = \nabla \cdot (V \nabla \cdot V)$, $\partial_i V_j = \partial_j V_i$ and

$$\nabla \cdot (V \nabla \cdot V) = \nabla \cdot \nabla \cdot (V \otimes V) - \frac{1}{2}\Delta(|V|^2). \tag{3.10}$$

Define $\mathcal{J}_1 : \ \rho \mapsto \mathcal{J}_1(\rho)$, where

$$\mathcal{J}_1(\rho) = e^{\mu_{en}t\Delta}\rho_0 + \mu_{in} \int_0^t e^{\mu_{en}(t-\tau)\Delta} \big[\nabla \cdot \nabla \cdot (V \otimes V) - \frac{1}{2}\Delta(|V|^2)\big] d\tau. \tag{3.11}$$

Next we will prove the a priori estimate of $\mathcal{J}_1(\rho)$.

Proposition 3.5. *Let $\mathcal{J}_1$ be as in (3.11), $(\mu_{en}, \mu_{in}) \in (0, \infty)^2$ and $\rho_0 \in BMO^{-2}$. Then we have*

$$\big\|\nabla \frac{1}{\Delta}\mathcal{J}_1(\rho)\big\|_{\mathcal{E}} \leq c\,\|\rho_0\|_{BMO^{-2}} + c_N \big\|V\big\|^2_{\mathcal{E}} \tag{3.12}$$

where $V = \nabla\frac{1}{\Delta}\rho$.

Proof. It suffices to show $\|e^{\mu_{en}t\Delta}\nabla\frac{1}{\Delta}\rho_0\|_{\mathcal{E}} \sim \|\nabla\frac{1}{\Delta}\rho_0\|_{BMO^{-1}} \sim \|\rho_0\|_{BMO^{-2}}$, which follows from [21, Lemma 16.1, p. 160] and Definition 1.4.

Estimate of $\int_0^t e^{\mu_{en}(t-\tau)\Delta}\nabla\frac{1}{\Delta}\big[\nabla\cdot\nabla\cdot(V\otimes V)-\frac{1}{2}\Delta(|V|^2)\big]d\tau$ follows from Lemma 3.3. Hence we finish the proof. □

Proof of Theorem 1.11. We divide the proof into three steps. At first, Proposition 3.1 ensures that $\mathcal{J}_1$ maps a closed ball $\overline{B(0;\varepsilon)}$ of $\nabla\frac{1}{\Delta}\mathcal{E} := \big\{V/\ \nabla\frac{1}{\Delta}V \in \mathcal{E}\big\}$ with $\varepsilon < 1/(4cc_N)$ into itself. Next, suppose ρ_1 and ρ_2 are two solutions of (3.6) with the same initial data $\rho_0 \in \overline{B(0;\varepsilon)}$, then

$$\|\mathcal{J}_1(\rho_1)-\mathcal{J}_1(\rho_2)\|_{\nabla\frac{1}{\Delta}\mathcal{E}} \le 4cc_N\varepsilon\|\rho_1-\rho_2\|_{\nabla\frac{1}{\Delta}\mathcal{E}}$$

where $4cc_{N,s}\varepsilon < 1$. Thus existence and uniqueness of solutions follow immediately from contraction arguments. It is worth mentioning that the time continuity fails due to the lack of density of $\mathcal{S}$ in BMO^{-2}. Conclusively, we finish the proof of Theorem 1.11. □

3.4. Embeddings for the case $s = 1$

In this subsection, we study several imbedding relations. Recall that in [14], the author proved that: if $N \in \mathbb{N}\cap[2,\infty)$ and $p \ge \frac{N}{2}$, then $L_x^{\frac{N}{2}} \subset \dot{B}^{\frac{N}{p}-2}_{p,\infty}$; if $N \ge 4$ and $p \ge 2$, then $\mathcal{PM}^{N-2} \subset \dot{B}^{-2+\frac{N}{p}}_{p,\infty}$, where

$$\mathcal{PM}^{N-2} = \big\{f\ /\ \sup_{\xi\in\mathbb{R}^N}|\xi|^{N-2}|\widehat{f}(\xi)| < \infty\big\};$$

if $N \ge 2$ and $p \in [\frac{N}{2},\infty]$, then $\dot{B}^0_{\frac{N}{2},2} \subset \dot{B}^{\frac{N}{p}-2}_{p,\infty}$. The proof is a direct consequence of Bernstein's inequalities (cf. [31]).

It remains to show that for any $N \ge 2$ and $p \in [1,\infty]$, $\dot{B}^{\frac{N}{p}-2}_{p,\infty} \subset BMO^{-2}$ and $\mathcal{B}^{-2}_2 \subset BMO^{-2}$, where

$$\dot{\mathcal{B}}^{-2}_2 = \big\{f\ /\ \|f\|_{\dot{\mathcal{B}}^{-2}_2} = \big(\sum\nolimits_k 2^{-4k}\|\psi_k\widehat{f}\|^2_{L^1_\xi}\big)^{\frac{1}{2}} < \infty\big\}.$$

Indeed,

$$\begin{aligned}\|u\|_{BMO^{-2}} &= \|e^{t\Delta}\nabla K_2 * u\|_{L^2_{uloc}} \le c_{N,p}\sup_{t>0} t^{\frac{1}{2}-\frac{N}{2p}}\|e^{t\Delta}\nabla K_2 * u\|_{L^p_x}\\ &\le c_{N,p}\|\nabla K_2 * u\|_{\dot{B}^{\frac{N}{p}-1}_{p,\infty}} \le c_{N,p}\|u\|_{\dot{B}^{\frac{N}{p}-2}_{p,\infty}}\end{aligned}$$

since $\nabla K_2 *$ is bounded from $\dot{B}^{\frac{N}{p}-2}_{p,\infty}$ to $\dot{B}^{\frac{N}{p}-1}_{p,\infty}$. It remains to show that for any $1 \le p_1 < p_2 \le \infty$ and $N \ge 2$, $\dot{B}^{\frac{N}{p_1}-2}_{p_1,\infty} \subset \dot{B}^{\frac{N}{p_2}-2}_{p_2,\infty}$, which is also a consequence of Bernstein's inequalities. At last, it is easy to prove that $\mathcal{B}^{-2}_2 \subset BMO^{-2}$ by making use of the Hausdorff–Young's inequality.

3.5. Analysis of (1.1) with $s = 2$ and $N = 4$

In this subsection, we study the following system

$$\rho_t - \mu_{en}\Delta\rho - \mu_{in}\nabla\cdot(\rho\nabla\Lambda^{-2s}\rho) = 0, \quad \rho|_{t=0} = \rho_0. \qquad (GD_{s=2})$$

Define

$$\mathcal{J}_2 : \rho \mapsto \mathcal{J}_2(\rho) = e^{\mu_{en}t\Delta}\rho_0 + \mu_{in}\int_0^t e^{\mu_{en}(t-\tau)\Delta}\nabla\cdot(\rho\nabla\Lambda^{-4}\rho)d\tau. \qquad (3.13)$$

Next we will prove the a priori estimate of $\mathcal{J}_2(\rho)$.

Proposition 3.6. *Let $\mathcal{J}_2$ be as in (3.13) and $N = 4$. For any $\mu_{en}, \mu_{in} > 0$ and $\rho_0 \in \dot{B}^{-3}_{4,2}$, we get*

$$\|\mathcal{J}_2(\rho)\|_{\widetilde{L}^\infty_t(\dot{B}^{-3}_{4,2})\cap\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})} \le c\,\|\rho_0\|_{\dot{B}^{-3}_{4,2}} + c\,\|\rho\|^2_{\widetilde{L}^\infty_t(\dot{B}^{-3}_{4,2})\cap\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})}. \qquad (3.14)$$

Additionally, $e^{\mu_{en}t\Delta}\rho_0 \to \rho_0$ in $\dot{B}^{-3}_{4,2}$ as $t \downarrow 0$.

Proof. Since $\|e^{\mu_{en}t\Delta}\rho_0\|_{\widetilde{L}^\infty_t(\dot{B}^{-3}_{4,2})\cap\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})} \le c\|\rho_0\|_{\dot{B}^{-3}_{4,2}}$ follows by standard argument, it suffices to control the remained part. Following the similar arguments as in [9, Lemmas 2.3 and 2.4], it is a direct consequence of [7, Lemma 2.1 on smoothing effect], $\dot{B}^{-2+k}_{2,2} \subset \dot{B}^{-3+k}_{4,2} \subset \dot{B}^{-4+k}_{\infty,2}$ for $k = 0, 1$ in 4 dimensional space, and Cauchy–Schwarz inequality in ℓ^1, we get

$$\begin{aligned}
&\|\int_0^t e^{\mu_{en}(t-\tau)\Delta}\nabla\cdot(\rho\nabla\Lambda^{-4}\rho)d\tau\|_{\widetilde{L}^\infty_t(\dot{B}^{-3}_{4,2})\cap\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})} \\
&\le c\,\min\left\{\|\rho\nabla\Lambda^{-4}\rho\|_{\widetilde{L}^2_t(\dot{B}^{-3}_{4,2})},\ \|\rho\nabla\Lambda^{-4}\rho\|_{\widetilde{L}^1_t(\dot{B}^{-2}_{4,2})}\right\} \\
&\le c\,\|\Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\|_{\widetilde{L}^2_t(\dot{B}^{-3}_{4,2})} + \|\rho\nabla\Lambda^{-4}\rho - \Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\|_{\widetilde{L}^1_t(\dot{B}^{-2}_{4,2})} \\
&\le c\,\|\Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\|_{\widetilde{L}^2_t(\dot{B}^{-3}_{4,2})} + \|\rho\nabla\Lambda^{-4}\rho - \Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\|_{\widetilde{L}^1_t(\dot{B}^{-1}_{2,2})} \\
&\le c\,\|\Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\|_{\widetilde{L}^2_t(\dot{B}^{-3}_{4,2})} + \|\Lambda^{-1}\big(\rho\nabla\Lambda^{-4}\rho - \Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\big)\|_{L^1_tL^2} \\
&\le c\,\|\rho\|_{\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-4}_{\infty,2})} + \|\rho\|^2_{\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})} \\
&\le c\,\|\rho\|_{\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})}\|\rho\|_{\widetilde{L}^\infty_t(\dot{B}^{-3}_{4,2})} + \|\rho\|^2_{\widetilde{L}^2_t(\dot{B}^{-2}_{4,2})},
\end{aligned}$$

where in the second term of the fourth inequality, (2.8) plays a key role in balancing Λ^{-1}. In fact, estimating $\Lambda^{-1}\big(\rho\nabla\Lambda^{-4}\rho - \Pi^{hl}_l(\rho,\nabla\Lambda^{-4}\rho)\big)$ is equivalent to estimate $\Pi^{hh}_l(\Lambda^{-2}\rho,\Lambda^{-2}\rho)$, $\Pi^{hh}_h(\Lambda^{-2}\rho,\Lambda^{-2}\rho)$ and $\Pi^{lh}_h(\rho,\Lambda^{-4}\rho)$. $\square$

Proof of Theorem 1.12. We divide the proof into three steps. At first, Proposition 3.6 ensures that $\mathcal{J}_2$ maps a closed ball $\overline{B(0;\varepsilon)}$ of $\widetilde{L}_t^\infty(\dot{B}_{4,2}^{-3}) \cap \widetilde{L}_t^2(\dot{B}_{4,2}^{-2})$ with $\varepsilon < 1/(4c^2)$ into itself. Hence $\mathcal{J}_2$ is well defined. Next, suppose ρ_1 and ρ_2 are two solutions of (3.13) with the same initial data $\rho_0 \in \overline{B(0;\varepsilon)}$, then

$$\|\mathcal{J}_2(\rho_1) - \mathcal{J}_2(\rho_2)\|_{\widetilde{L}_t^\infty(\dot{B}_{4,2}^{-3})\cap\widetilde{L}_t^2(\dot{B}_{4,2}^{-2})} \leq 4c^2\varepsilon\|\rho_1 - \rho_2\|_{\widetilde{L}_t^\infty(\dot{B}_{4,2}^{-3})\cap\widetilde{L}_t^2(\dot{B}_{4,2}^{-2})}$$

where $4c^2\varepsilon < 1$. Indeed, denote $\rho_j^{(4)} = \Lambda^{-4}\rho_j$. Then we have

$$2\left(\rho_1\nabla\rho_1^{(4)} - \rho_2\nabla\rho_2^{(4)}\right) = (\rho_1+\rho_2)\nabla(\rho_1-\rho_2)^{(4)} + (\rho_1-\rho_2)\nabla(\rho_1+\rho_2)^{(4)}.$$

It is clear that the right hand side of the above identity is symmetric and satisfies Lemmas 2.1 and 2.4. Thus existence and uniqueness of solutions follow immediately from contraction arguments. At last, time continuity follows from Proposition 3.6. Conclusively, we finish the proof of Theorem 1.12. □

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