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Endpoint bilinear estimates and applications to the two-dimensional Poisson–Nernst–Planck system

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Abstract

We study the Cauchy problem of the two-dimensional Poisson–Nernst–Planck (PNP) system in Besov spaces $\dot{B}_4^{-3/2,r}$ for $r \geqslant 2$. Our work shows a *dichotomy* of well-posedness and ill-posedness depending only on r. Specifically, when $r = 2$, combining the key bilinear estimates in $L_T^2\dot{\mathcal{W}}^{-1/2,4} \cap L_T^4\dot{\mathcal{W}}^{-1,4}$ with the heat semigroup characterization of Besov spaces, we prove the well-posedness of the PNP in $\dot{B}_4^{-3/2,2}$, while for $r > 2$ we show that the PNP is ill-posed in $\dot{B}_4^{-3/2,r}$ in the sense that the difference of the charges must satisfy certain requirements, i.e. either the difference belongs to $\dot{B}_4^{-3/2,r}$ for $r > 2$ and the summation belongs to $\dot{B}_4^{-3/2,2}$, or the difference belongs to $\dot{B}_4^{-3/2,2}$ and the summation belongs to $\dot{B}_4^{-3/2,r}$ for $r > 2$. Thus our results indicate that the difference of charges plays a crucial role and might provide some instability criterion for numerical analysis.

Mathematics Subject Classification: 35K45, 42B37

1. Introduction

In this article, we study the following two-dimensional (2D) normalized Poisson–Nernst–Planck (PNP) system

$$\begin{cases} \partial_t v = \Delta v - \nabla \cdot \left(w \dfrac{1}{\Delta} \nabla v \right), \\ \partial_t w = \Delta w - \nabla \cdot \left(v \dfrac{1}{\Delta} \nabla v \right) \end{cases} \tag{1}$$

with $(t, x) \in (0, \infty) \times \mathbb{R}^2$, $\frac{1}{\Delta}$ being the Fourier multiplier of symbol $\frac{1}{-|\xi|^2}$, i.e. $\frac{1}{\Delta}u = \mathcal{F}^{-1}(\frac{1}{-|\xi|^2}\mathcal{F}u(\xi))$, and initial data (v_0, w_0).

It is clear that system (1) is derived from

$$\begin{cases} n_t = \nabla \cdot (\nabla n - n\nabla\phi), \\ p_t = \nabla \cdot (\nabla p + p\nabla\phi), \\ \Delta\phi = n - p \end{cases} \tag{2}$$

by setting $v = n - p$ and $w = n + p$. Systems (1) and (2) appear in the context as the Nernst–Plank equation in astronomy in [2]. It is called the Van Roosbroeck system in semiconductor devices in [30], and the Debye–Hückel system modelling the diffusion of ions in an electrolyte in [6]. It is worth mentioning that the Keller–Segel system modelling two species [33] is similar to (1). For other related works, we refer readers to [1, 3, 7, 10, 17, 19–21, 26–29, 32] and the references therein.

Besides the abundant numerical results, mathematical analysis for the PNP system has been studied by many authors. In 1970s, Mock [22, 23] proved the solvability of the steady-state problems, the global existence and uniqueness result and exponential decay for the instationary problem. More general global existence and uniqueness results were proved by Gajewski [13], Gajewski and Gröger [14] by maximum principle, compactness arguments and iteration technique.

Formally, if we let $v \equiv w$, then the PNP system (1) is reduced to

$$v_t = \nabla \cdot \left(\nabla v - v\frac{1}{\Delta}\nabla v\right), \tag{3}$$

which nonlinear term is similar to the elliptic–parabolic Keller–Segel system, see for instance [15].

As we know the critical function spaces for initial data should be invariant under

$$(v_0(x), w_0(x)) \to (\lambda^2 v_0(\lambda x), \lambda^2 w_0(\lambda x)). \tag{4}$$

Particularly, in the 2D case ($n = 2$), we get the following invariant spaces

$$\mathcal{H}^1 \hookrightarrow \dot{B}_1^{0,2} \hookrightarrow \dot{B}_4^{-\frac{3}{2},r} \hookrightarrow \dot{B}_q^{-2+\frac{2}{q},\infty} \hookrightarrow BMO^{-2} \hookrightarrow \dot{B}_\infty^{-2,\infty} \text{ for } 4 < q < \infty, r \geqslant 2.$$

Recently, Ogawa and Shimizu [24, 25] studied the well-posedness for the system (1) in Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and the well-posedness for the system (3) in Besov space $\dot{B}_1^{0,2}(\mathbb{R}^2)$. Zhao *et al* [34] proved the global well-posedness of (1) in $\dot{B}_q^{s,r}(\mathbb{R}^n)$ for $n \geqslant 2$, $1 \leqslant r \leqslant \infty$, $s > -\frac{3}{2}$ and $q = \frac{n}{s+2}$. Iwabuchi [16] proved the global well-posedness of the Keller–Segel system in $\dot{B}_q^{s,\infty}(\mathbb{R}^n)$ for $n \geqslant 1$, $1 < q < \infty$ and $s = -2 + \frac{n}{q} > -2$. Deng and Liu [8] proved the global well-posedness for the general diffusion system. In particular, when $s = 1$, the general diffusion system is similar to the Keller–Segel system (3), and it is well-posed in BMO^{-2} since

$$v\frac{1}{\Delta}\nabla v = \nabla \cdot \left(\frac{1}{\Delta}\nabla v \otimes \frac{1}{\Delta}\nabla v\right) - \frac{1}{2}\nabla\left(\frac{1}{\Delta}\nabla v \cdot \frac{1}{\Delta}\nabla v\right). \tag{5}$$

The previous works for equations (1)–(3) and (5) suggest that

- $s = -2$ is the optimal regular index for the Keller–Segel system (3);
- $s = -\frac{3}{2}$ seems to be the optimal regular index for the PNP system (1).

The main purpose of this paper is to give an affirmative answer to the optimality of $s = -\frac{3}{2}$ for the 2D PNP system. Precisely,

- for any $-\frac{3}{2} < s \leqslant 0$, the system (1) is globally well-posed in $\dot{B}_q^{s,\infty}(\mathbb{R}^2)$ for $q = \frac{2}{s+2}$, see for instance [34];

- for $s = -\frac{3}{2}$, we prove that the system (1) is well-posed in $\dot{B}_4^{-3/2,2}(\mathbb{R}^2)$ and ill-posed in $\dot{B}_4^{-3/2,r}(\mathbb{R}^2)$ for $r > 2$;
- for $-2 \leqslant s < -\frac{3}{2}$, it is believed that the system (1) is ill-posed even in $\dot{B}_q^{s,1}(\mathbb{R}^2)$ for $q = \frac{2}{s+2}$.

According to our proof of well-posedness (theorem 1.3), to establish a complete *dichotomy* of well-posedness and ill-posedness in $\dot{B}_4^{-3/2,r}(\mathbb{R}^2)$ depending on $1 \leqslant r \leqslant \infty$, it suffices to prove well-posedness for system (1) in $\dot{B}_4^{-3/2,1}(\mathbb{R}^2)$. However, it seems rather difficult to achieve this goal. Meanwhile, it also seems difficult to apply our proof of the well-posedness of the higher dimensional cases $n \geqslant 3$. Therefore, it should be an interesting problem to prove the well-posedness of the PNP system in $\dot{B}_{2n}^{-\frac{3}{2},r}(\mathbb{R}^n)$ for $1 \leqslant r \leqslant 2$ and ill-posedness of the PNP system in $\dot{B}_{2n}^{-\frac{3}{2},r}$ for $r > 2$ and $n \geqslant 2$.

To explore the difference of $s = -\frac{3}{2}$ and $s = -2$, we first study the differences of systems (1) and (3). It is easy to check that the bilinear term $v\frac{1}{\Delta}\nabla v$ in (1) and (3) satisfies identity (5). However, $w\frac{1}{\Delta}\nabla v$ does not satisfies (5). This shows that the PNP system and the Keller–Segel system have *essentially different structures*. It is worth mentioning that our results (see theorems 1.4 and 1.5) show that only v (i.e. the *difference* of the charges) produces ill-posedness.

Since the proof is formulated by a dyadic decomposition, let us briefly explain how it may be built in $\mathbb{R}^2$, see for instance, [31]. Let $\varphi(\xi) = \varphi(|\xi|)$ be a smooth function valued in [0, 1] such that

$$\mathrm{supp}\varphi \subset \left\{\xi \in \mathbb{R}^2;\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\right\} \qquad \text{and} \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \qquad \xi \neq 0. \tag{6}$$

For any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$, we define the homogeneous dyadic block and partial summation operator as follows

$$\Delta_j f(x) := \mathcal{F}_\xi^{-1}(\varphi(2^{-j}\xi)\widehat{f}(\xi))(x), \qquad S_j f(x) := \sum_{i\leqslant j-1} \Delta_i f(x) \quad \text{for all } j \in \mathbb{Z}. \tag{7}$$

The Littlewood–Paley decomposition satisfies the quasi-orthogonal properties:

$$\Delta_i\Delta_j f \equiv 0 \quad \text{if } |i-j| \geqslant 2, \qquad \Delta_j(S_{i-1} f \Delta_i g) \equiv 0 \quad \text{if } |i-j| \geqslant 5. \tag{8}$$

Using Bony's decomposition, one can split the product of two functions as follows:

$$fg = T_f g + T_g f + R(f, g), \tag{9}$$

where $T_f g = \sum_j S_{j-1} f \Delta_j g$, $T_g f = \sum_j S_{j-1} g \Delta_j f$ and $R(f, g) = \sum_j \sum_{\ell=-1}^{1} \Delta_{j+\ell} f \Delta_j g$.

In order to exclude nonzero polynomials, it is natural to use $Z'(\mathbb{R}^2)$ to denote the subset of tempered distributions $f \in S'(\mathbb{R}^2)$ modulo all polynomials set $P(\mathbb{R}^2)$, i.e. $Z'(\mathbb{R}^2) = S'(\mathbb{R}^2)/P(\mathbb{R}^2)$.

Next we list the definitions of Besov space/Triebel–Lizorkin space (see [31]) and fractional Sobolev spaces $\dot{\mathcal{W}}^{s,q}(\mathbb{R}^2)$.

Definition 1.1. *For $(s, q, r) \in \mathbb{R} \times (1, \infty) \times [1, \infty]$, we define $\dot{B}_q^{s,r}(\mathbb{R}^2)$, $\dot{F}_q^{s,r}(\mathbb{R}^2)$ and $\dot{\mathcal{W}}^{s,q}(\mathbb{R}^2)$ to be the set of distributions f in $\mathcal{S}_h'(\mathbb{R}^2)$ satisfying*

$$\|f\|_{\dot{B}_q^{s,r}(\mathbb{R}^2)} = \|\{2^{js}\|\Delta_j f\|_{L^q(\mathbb{R}^2)}\}_{j\in\mathbb{Z}}\|_{\ell^r} < \infty, \tag{10}$$

$$\|f\|_{\dot{F}_q^{s,r}(\mathbb{R}^2)} = \|\|\{2^{js}\Delta_j f\}_{j\in\mathbb{Z}}\|_{\ell^r}\|_{L^q(\mathbb{R}^2)} < \infty, \tag{11}$$

$$\|f\|_{\dot{\mathcal{W}}^{s,q}(\mathbb{R}^2)} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^q(\mathbb{R}^2)} < \infty, \tag{12}$$

respectively.

Applying the classical multiplier theorem, for any $1 < q < \infty$ one can prove that

$$\dot{F}_q^{s,2}(\mathbb{R}^2) = \dot{\mathcal{W}}^{s,q}(\mathbb{R}^2) \qquad \text{and} \qquad \dot{F}_2^{s,2}(\mathbb{R}^2) = \dot{B}_2^{s,2}(\mathbb{R}^2) = \dot{\mathcal{W}}^{s,2}(\mathbb{R}^2). \tag{13}$$

Definition 1.2. *For $(s, \varrho, q, T) \in \mathbb{R} \times [1, \infty]^2 \times (0, \infty)$, we denote $L_T^\varrho \dot{\mathcal{W}}^{s,q}$ to be the set of functions f such that*

$$\|f\|_{L_T^\varrho \dot{\mathcal{W}}^{s,q}} = \|(-\Delta)^{\frac{s}{2}} f\|_{L_T^\varrho L^q} < \infty. \tag{14}$$

Notations. We define several notations which will be used throughout this paper:

- we shall use C and c to denote universal constants which may change from line to line and we denote $A \leqslant CB$ by $A \lesssim B$ and $A \lesssim B \lesssim A$ by $A \sim B$;
- $\mathcal{F}f$ and $\widehat{f}$ stand for Fourier transform of f with respect to space variable, while $\mathcal{F}^{-1}$ stands for the inverse Fourier transform;
- let $\Lambda := \sqrt{-\Delta}$ and $\nabla^{-1} = -\frac{1}{\Delta}\nabla$;
- for any $1 \leqslant \varrho \leqslant \infty$, we denote $L^\varrho(0, T)$, $L^\varrho(T_1, T_2)$ and $L^\varrho(0, \infty)$ by L_T^ϱ, $L_{[T_1,T_2]}^\varrho$ and L_t^ϱ, respectively;
- we also use $\dot{B}_q^{s,r}$, $\dot{F}_q^{s,r}$ and $\dot{\mathcal{W}}^{s,q}$ to denote $\dot{B}_q^{s,r}(\mathbb{R}^2)$, $\dot{F}_q^{s,r}(\mathbb{R}^2)$ and $\dot{\mathcal{W}}^{s,q}(\mathbb{R}^2)$ if there is no confusion about the domain, and similar conventions are applied;
- $X_{[T_\alpha,T_{\alpha+1}]} = L^4_{[T_\alpha,T_{\alpha+1}]}\dot{\mathcal{W}}^{-1,4} \cap L^2_{[T_\alpha,T_{\alpha+1}]}\dot{\mathcal{W}}^{-\frac{1}{2},4}$, $X_T = L_T^4\dot{\mathcal{W}}^{-1,4} \cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}$ and $X_\infty = L_t^2\dot{\mathcal{W}}^{-\frac{1}{2},4} \cap L_t^4\dot{\mathcal{W}}^{-1,4}$.

Now we are ready to state our main results.

Theorem 1.3. *For any initial data $(v_0, w_0) \in \dot{B}_4^{-3/2,2} \times \dot{B}_4^{-3/2,2}$, there exists positive $T = T(v_0, w_0)$ such that*

$$\|e^{t\Delta}v_0\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-1/2,4}} + \|e^{t\Delta}w_0\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-1/2,4}}$$

is small and system (1) has a unique local solution (v, w) satisfying

$$v,\ w \in C([0, T]; \dot{B}_4^{-3/2,2}) \cap L_T^4\dot{\mathcal{W}}^{-1,4} \cap L_T^2\dot{\mathcal{W}}^{-1/2,4}. \tag{15}$$

Furthermore, if $\|v_0\|_{\dot{B}_4^{-3/2,2}} + \|w_0\|_{\dot{B}_4^{-3/2,2}}$ is suitably small, then system (1) has a unique global solution satisfying

$$v,\ w \in C([0, \infty); \dot{B}_4^{-3/2,2}) \cap L_t^2\dot{\mathcal{W}}^{-1/2,4} \cap L_t^4\dot{\mathcal{W}}^{-1,4}. \tag{16}$$

Remark 1. Assumption of $(v_0, w_0) \in \dot{B}_4^{-3/2,2} \times \dot{B}_4^{-3/2,2}$ is a natural consequence of [18, theorem 5.4], isomorphism and $\|e^{t\Delta}f\|_{L_t^2\dot{\mathcal{W}}^{-\frac{1}{2},4}\cap L_t^4\dot{\mathcal{W}}^{-1,4}} \sim \|f\|_{\dot{B}_4^{-3/2,2}}$.

The next results are the ill-posedness of the system (1) in Besov spaces.

Theorem 1.4. *For any $\delta > 0$ and $r > 2$, there exists a solution (v, w) to system (1) with initial data $(v_0, w_0) \in \dot{B}_4^{-3/2,r} \times \dot{B}_4^{-3/2,2}$ satisfying*

$$\|v_0\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim \delta \qquad \text{and} \qquad \|w_0\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim \delta$$

such that for some $0 < T < \delta$,

$$\|v(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} \gtrsim \frac{1}{\delta} \qquad \text{but} \qquad \|w(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim \delta.$$

Theorem 1.5. *For any* $\delta > 0$ *and* $r > 2$, *there exists a solution* (v, w) *to system (1) with initial data* $(v_0, w_0) \in \dot{B}_4^{-3/2,2} \times \dot{B}_4^{-3/2,r}$ *satisfying*

$$\|v_0\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim \delta \qquad \text{and} \qquad \|w_0\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim \delta$$

such that for some $0 < T < \delta$,

$$\|v(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} \gtrsim \frac{1}{\delta} \qquad \text{but} \qquad \|w(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim \delta.$$

Remark 2. From theorems 1.4 and 1.5, it is clear that v is of great importance in the study of ill-posedness. There are several explanations. On the one hand, the nonlinear term of w is $v\frac{1}{\Delta}\nabla v$ satisfying (5) and hence is better than $w\frac{1}{\Delta}\nabla v$; on the other hand, from (2) we see that $v = n - p$ is naturally related to the potential ϕ and the bilinear force terms.

This paper is organized as follows. In section 2, we establish the key bilinear estimates; in section 3, we prove the well-posedness; in section 4, we first construct a very special initial data and prove some necessary estimates about the first and second approximation terms which will be used in controlling the remainder term. Finally, combining all the *a priori* estimates we prove the ill-posedness.

2. Endpoint bilinear estimates

In this section, we will prove well-posedness of system (1). As usual, we rewrite system (1) into the equivalent integral equations:

$$v = e^{t\Delta}v_0 + \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(w\nabla^{-1}v)\,d\tau, \tag{17}$$

$$w = e^{t\Delta}w_0 + \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(v\nabla^{-1}v)\,d\tau, \tag{18}$$

where $e^{t\Delta}v_0 = (2\pi t)^{-\frac{n}{2}}e^{-\frac{|\cdot|^2}{4t}} * v_0(\cdot)$ and similar convention is applied for $e^{t\Delta}w_0$.

For simplicity, later on we denote

$$B(w, v) = \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(w\nabla^{-1}v)\,d\tau. \tag{19}$$

2.1. Preliminary lemmas

In what follows, we will use the well-known vector-valued maximal inequality proved in [11].

Lemma 2.1. *Let* $(r, q) \in (1, \infty] \times (1, \infty)$ *or* $r = q = \infty$. *Suppose that* $\{f_j\}_{j\in\mathbb{Z}}$ *is a sequence of functions in* L^q *with property that* $\|\{f_j\}_j\|_{\ell^r} \in L^q$. *Then*

$$\left\|\left\{Mf_j\right\}_j\right\|_{L^q\ell^r} \lesssim \left\|\left\{|f_j|\right\}_j\right\|_{L^q\ell^r},$$

with $B_R(x) = \{y \in \mathbb{R}^n;\ |x - y| < R\}$ *and*

$$(Mf)(x) = \sup_{R>0}\frac{1}{|B_R(x)|}\int_{B_R(x)}|f(y)|\,dy. \tag{20}$$

Applying lemma 2.1 and following the similar argument as in [5, estimates for I and II, pp 660–1], we have the following lemma.

Lemma 2.2. *For any two functions f and g, recall that $T_f g = \sum_{j\in\mathbb{Z}} S_{j-1} f \Delta_j g$. Then for any $(s, \sigma, q, q_1, q_2) \in \mathbb{R} \times [0, \infty) \times (1, \infty)^3$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, we have*

$$\|T_f g\|_{\dot{\mathcal{W}}^{s,q}} \lesssim \|f\|_{\dot{\mathcal{W}}^{-\sigma,q_1}} \|g\|_{\dot{\mathcal{W}}^{s+\sigma,q_2}}.$$

Proof. By successive applications of (13), (11), (8)–(9), lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
\|T_f g\|_{\dot{\mathcal{W}}^{s,q}} &\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} 2^{2sj} \left| \Delta_j \left(\sum_{j'} S_{j'-1} f \Delta_{j'} g \right) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{|j-j'|\leqslant 4} 2^{2sj} |M(S_{j'-1} f \Delta_{j'} g)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \\
&\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} \sum_{|j-j'|\leqslant 4} 2^{2sj} |S_{j'-1} f \, \Delta_{j'} g|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \\
&\lesssim \left\| \left\{ 2^{-\sigma j'} |S_{j'-1} f| \right\}_{j'} \right\|_{L^{q_1} \ell^\infty} \left\| \left\{ 2^{(s+\sigma) j'} |\Delta_{j'} g| \right\}_{j'} \right\|_{L^{q_2} \ell^2} \\
&\lesssim \|f\|_{\dot{F}^{-\sigma,2}_{q_1}} \|g\|_{\dot{F}^{s+\sigma,2}_{q_2}},
\end{aligned}$$

where $\|\{2^{-\sigma j'}|S_{j'-1} f|\}_{j'}\|_{L^{q_1}\ell^\infty} \lesssim \|f\|_{\dot{F}^{-\sigma,2}_{q_1}}$. □

The next lemma is about the maximal regularity for the following heat equation whose proof is rather simple via energy method.

$$\begin{cases} u_t - \Delta u = \Lambda f, \\ u|_{t=0} = 0. \end{cases} \tag{21}$$

We refer readers to [18, chapter 7] to see the general $L^p_T L^q$ regularity theorem.

Lemma 2.3. *Let $T > 0$ and $n = 2$. For any function $f \in L^1_T \dot{\mathcal{W}}^{1,2} + L^2_T L^2$ with norm $\min_{f=f_1+f_2} \|\nabla f_1\|_{L^1_T L^2} + \|f_2\|_{L^2_T L^2}$, we define $\mathcal{A} : f(x,t) \mapsto \int_0^t e^{(t-\tau)\Delta} \Delta f(x,\tau)\, d\tau$. Then there exists a positive constant C depending only on dimension n such that*

$$\|\mathcal{A} f\|_{L^\infty_T \dot{\mathcal{W}}^{-1,2}} + \|\mathcal{A} f\|_{L^2_T L^2} \leqslant C \|f\|_{L^2_T L^2 + L^1_T \dot{\mathcal{W}}^{1,2}}.$$

Proof. The above result is not new but for reader's convenience, we give a short proof. By classical energy method and (21): $\Lambda u = -\mathcal{A} f$ and $u_0 = 0$, we observe that for $0 < t < T$,

$$\|\mathcal{A} f\|_{L^2_T L^2} \leqslant C \|\nabla u\|_{L^2_T L^2}, \qquad \|\mathcal{A} f\|_{L^\infty_T \dot{\mathcal{W}}^{-1,2}} \leqslant C \|u\|_{L^\infty_T L^2}$$

and

$$\begin{aligned}
\frac{1}{2}\|u(t)\|^2_{L^2} + \|\nabla u\|^2_{L^2_T L^2} &\leqslant C \min\{\|\Lambda^{-1} \nabla f \cdot \nabla u\|_{L^1_T L^1}, \ \|\Lambda f u\|_{L^1_T L^1}\} \\
&\leqslant C \|f\|^2_{L^2_T L^2 + L^1_T \dot{\mathcal{W}}^{1,2}} + \frac{1}{4}(\|\nabla u\|^2_{L^2_T L^2} + \|u\|^2_{L^\infty_T L^2}),
\end{aligned}$$

which gives the desired estimate. □

2.2. Bilinear estimates

Next we prove the following key bilinear estimates.

Lemma 2.4. *Recall that $X_T = L^4_T\dot{\mathcal{W}}^{-1,4} \cap L^2_T\dot{\mathcal{W}}^{-\frac{1}{2},4}$. Let $B(v, w)$ be defined as in (19), $T > 0$ and $n = 2$. Then we have*

$$\|B(w, v)\|_{L^\infty_T\dot{B}_4^{-\frac{3}{2},2}} + \|B(w, v)\|_{X_T} \lesssim \|w\|_{X_T}\|v\|_{X_T}.$$

Proof. From $\dot{\mathcal{W}}^{-1,2} \hookrightarrow \dot{B}_4^{-\frac{3}{2},2}$, $\dot{\mathcal{W}}^{-\frac{1}{2},2} \hookrightarrow \dot{\mathcal{W}}^{-1,4}$ and $L^2 \hookrightarrow \dot{\mathcal{W}}^{-\frac{1}{2},4}$, it suffices to estimate $\|B(w, v)\|_{L^\infty_T\dot{\mathcal{W}}^{-1,2}\cap L^2_T L^2}$. Using Bony's decomposition in (9), we get

$$w\nabla^{-1}v = T_w\nabla^{-1}v + T_{\nabla^{-1}v}w + R(w, \nabla^{-1}v).$$

Let $f = \Lambda^{-1}(w\nabla^{-1}v)$. We get $\mathcal{A}f = B(w, v)$. Splitting f into $f_1 = \Lambda^{-1}R(w, \nabla^{-1}v)$ and $f_2 = \Lambda^{-1}(T_w\nabla^{-1}v + T_{\nabla^{-1}v}w)$, then applying lemma 2.3 and the simple fact $L^4_T\dot{\mathcal{W}}^{-\frac{1}{2},2} = [L^\infty_T\dot{\mathcal{W}}^{-1,2}, L^2_T L^2]_{(\frac{1}{2})}$ to $B(w, v)$, we have

$$\begin{aligned}\|B(w, v)\|_{L^\infty_T\dot{B}_4^{-\frac{3}{2},2}} + \|B(w, v)\|_{L^4_T\dot{\mathcal{W}}^{-1,4}\cap L^2_T\dot{\mathcal{W}}^{-\frac{1}{2},4}} &\lesssim \|B(w, v)\|_{L^\infty_T\dot{\mathcal{W}}^{-1,2}\cap L^2_T L^2}\\ &\lesssim \|R(w, \nabla^{-1}v)\|_{L^1_T L^2} + \|T_w\nabla^{-1}v + T_{\nabla^{-1}v}w\|_{L^2_T\dot{\mathcal{W}}^{-1,2}}.\end{aligned}$$

From definition of $R(w, \nabla^{-1}v)$ in (9), Cauchy–Schwarz inequality, lemma 2.1, (13) and definition 1.1, we have

$$\begin{aligned}\|R(w, \nabla^{-1}v)\|_{L^1_T L^2} &\lesssim \left\| \|\{2^{-\frac{j}{2}}|\Delta_j w|\}_j\|_{\ell^2}\|\{2^{\frac{j}{2}}|\Delta_j\nabla^{-1}v|\}_j\|_{\ell^2}\right\|_{L^1_T L^2}\\ &\lesssim \left\|\|\{2^{-\frac{j}{2}}|\Delta_j w|\}_j\right\|_{L^2_T L^4\ell^2}\left\|\{2^{-\frac{j}{2}}|M(\Delta_j v)|\}_j\right\|_{L^2_T L^4\ell^2}\\ &\lesssim \left\|\|\{2^{-\frac{j}{2}}|\Delta_j w|\}_j\right\|_{L^2_T L^4\ell^2}\left\|\{2^{-\frac{j}{2}}|\Delta_j v|\}_j\right\|_{L^2_T L^4\ell^2}\\ &\lesssim \|w\|_{L^2_T\dot{\mathcal{W}}^{-\frac{1}{2},4}}\|v\|_{L^2_T\dot{\mathcal{W}}^{-\frac{1}{2},4}},\end{aligned}$$

where in the second inequality we used

$$|\Delta_j\nabla^{-1}v(x)| \lesssim \sum_{i=-1,0,1}|\Delta_{j+i}\nabla^{-1}\Delta_j v| \lesssim 2^{-j}M(\Delta_j v)(x).$$

By applying lemma 2.2 to $T_w\nabla^{-1}v$ with $(s, \sigma, q, q_1, q_2) = (-1, 1, 2, 4, 4)$ and to $T_{\nabla^{-1}v}w$ with $(s, \sigma, q, q_1, q_2) = (-1, 0, 2, 4, 4)$, we get

$$\begin{aligned}\|T_w\nabla^{-1}v\|_{L^2\dot{\mathcal{W}}^{-1,2}} &\lesssim \|w\|_{L^4_T\dot{\mathcal{W}}^{-1,4}}\|\nabla^{-1}v\|_{L^4_T L^4} \lesssim \|w\|_{L^4_T\dot{\mathcal{W}}^{-1,4}}\|v\|_{L^4_T\dot{\mathcal{W}}^{-1,4}},\\ \|T_{\nabla^{-1}v}w\|_{L^2\dot{\mathcal{W}}^{-1,2}} &\lesssim \|\nabla^{-1}v\|_{L^4_T L^4}\|w\|_{L^4_T\dot{\mathcal{W}}^{-1,4}} \lesssim \|v\|_{L^4_T\dot{\mathcal{W}}^{-1,4}}\|w\|_{L^4_T\dot{\mathcal{W}}^{-1,4}}.\end{aligned}$$

Combining the above estimates we obtain the desired results. □

3. Analysis of well-posedness

Before giving the proof, we recall the well-known Picard contraction principle, see for instance, [18, theorem 13.2, pp 124].

Lemma 3.1. *Let $(\mathcal{X} \times \mathcal{X}, \ \|\cdot\|_{\mathcal{X}} + \|\cdot\|_{\mathcal{X}})$ be an abstract Banach product space, and $B : \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ be a bilinear operator such that for any $(v, w) \in \mathcal{X}\times\mathcal{X}$, there exist positive constant c and if*

$$\|B(w, v)\|_{\mathcal{X}} + \|B(v, v)\|_{\mathcal{X}} \leqslant c\,(\|v\|_{\mathcal{X}}\|w\|_{\mathcal{X}} + \|v\|_{\mathcal{X}}\|v\|_{\mathcal{X}}),$$

then for any (v_0, w_0) *satisfying* $\|(e^{t\Delta}v_0, e^{t\Delta}w_0)\|_{\mathcal{X}\times\mathcal{X}} < 1/4c$*, the following system*

$$(v, w) = (e^{t\Delta}v_0, e^{t\Delta}w_0) + (B(w, v), B(v, v))$$

has a solution (v, w) *in* $\mathcal{X} \times \mathcal{X}$*. In particular, the solution is such that*

$$\|(v, w)\|_{\mathcal{X}\times\mathcal{X}} \leqslant 2\|(e^{t\Delta}v_0, e^{t\Delta}w_0)\|_{X\times X}$$

and is the only one such that $\|(v, w)\|_{\mathcal{X}\times\mathcal{X}} < 1/2c$.

Recall that $X_T = L^4_T\dot{\mathcal{W}}^{-1,4} \cap L^2_T\dot{\mathcal{W}}^{-\frac{1}{2},4}$ and $X_\infty = L^4_t\dot{\mathcal{W}}^{-1,4} \cap L^2_t\dot{\mathcal{W}}^{-\frac{1}{2},4}$. Now we divide the proof of theorem 1.3 into two parts: local and global well-posedness.

Proof of theorem 1.3. At first, applying lemmas 2.4 and 3.1, we prove that there exist $T > 0$ and a unique solution $(v, w) \in \overline{B(0, 2A_0)} \subset X_T \times X_T$ to system (1) if $A_0 := \|(e^{t\Delta}v_0, e^{t\Delta}w_0)\|_{X_T\times X_T} < 1/4c$.

Indeed, for any given initial $v_0 \in \dot{B}^{-3/2,2}_4$, from [18, theorem 5.3, pp 44] we get

$$\|e^{t\Delta}v_0\|_{X_\infty} \sim \|v_0\|_{\dot{B}^{-3/2,2}_4} < \infty.$$

Therefore, as the length of $[0, T] \subset [0, \infty)$ tends to zero, $\|e^{t\Delta}v_0\|_{X_T}$ is surely smaller than $1/4c$. A similar argument is applied for w_0. Thus the existence, uniqueness and continuity of the solution map follow from standard fixed point and dense arguments. Hence we omit the details.

Note that the bilinear estimates in lemma 2.4 can be extended to $T = \infty$. However, in this case, $\|e^{t\Delta}v_0\|_{X_\infty}$ and $\|e^{t\Delta}w_0\|_{X_\infty}$ are not necessarily small. Hence the smallness condition is needed for global well-posedness. The remaining part of the proof follows by applying a standard argument, see for instance [12]. □

4. Analysis of ill-posedness

In this section, we prove the 'norm inflation' of system (1) in[4] $\dot{B}^{-3/2,r}_4(\mathbb{T}^2)$ with $r > 2$ since following [9] we can also prove the whole space domain case. We rewrite the solution to the system (1) as a summation of the first approximation, second approximation and remainder:

$$v = v_1 + v_2 + y, \qquad w = w_1 + w_2 + z, \tag{22}$$

where

$$v_1 = e^{t\Delta}v_0, \quad v_2 = B(w_1, v_1), \qquad w_1 = e^{t\Delta}w_0 \quad \text{and} \quad w_2 = B(v_1, v_1). \tag{23}$$

Moreover, the remainder terms satisfy the following integral equations:

$$y = V_2 + V_1 + V_0, \qquad z = W_2 + W_1 + W_0, \tag{24}$$

on $(0, \infty)$ with the initial conditions $(y(0), z(0)) = (0, 0)$,

$$\begin{cases} V_2 = B(z, y), \\ V_1 = B(z, v_1 + v_2) + B(w_1 + w_2, y), \\ V_0 = B(w_1, v_2) + B(w_2, v_1 + v_2), \end{cases} \tag{25}$$

and

$$\begin{cases} W_2 = B(y, y), \\ W_1 = B(y, v_1 + v_2) + B(v_1 + v_2, y), \\ W_0 = B(v_1, v_2) + B(v_2, v_1 + v_2). \end{cases} \tag{26}$$

In the remaining part of this section, we will present the special construction of initial data and give some preliminary estimates for v_0, w_0, v_1, w_1, w_2 and for v_2. Then by establishing the upper bounds of y and z, we complete the proof of theorem 1.4. At last, following a similar argument, we sketch the proof of theorem 1.5.

[4] From now on, we omit the notation of domain, for instance, we denote $\dot{B}^{-3/2,r}_4(\mathbb{T}^2)$ by $\dot{B}^{-3/2,r}_4$.

4.1. Construction of the initial data and the related estimates

For large enough ρ and Q (will be specified later), we define the initial data as follows:

$$v_0(x) = \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} k_s^{\frac{3}{2}} \cos(k_s x_2), \qquad w_0(x) = \frac{1}{\sqrt{Q\rho}} \sum_{s=1}^{\rho} h_s^{\frac{3}{2}} \cos(h_s x_2), \tag{27}$$

where $x \in \mathbb{T}^2$, $k_s = 2^{\frac{(s+1)(s+2m_0)}{2}}$, $h_s = 1 + k_s$ and m_0 is a large number depending on δ.

According to the choice of k_s, we observe that v_0 and w_0 themselves can be regarded as summations of the corresponding Littlewood–Paley decompositions. Moreover, in each dyadic annulus, there exists at most one k_s. Therefore, for any $\sigma \in \mathbb{R}$ and $1 \leqslant r \leqslant \infty$, we can equivalently define

$$\|v_0\|_{\dot{B}_4^{\sigma,r}} = \frac{Q}{\sqrt{\rho}} \left\| \left\{ k_s^{\sigma+\frac{3}{2}} \|\cos(k_s x_2)\|_{L^4(\mathbb{T}^2)} \right\}_{s=1}^{\rho} \right\|_{\ell^r}, \tag{28}$$

$$\|w_0\|_{\dot{B}_4^{\sigma,r}} = \frac{Q}{\sqrt{\rho}} \left\| \left\{ h_s^{\sigma+\frac{3}{2}} \|\cos(h_s x_2)\|_{L^4(\mathbb{T}^2)} \right\}_{s=1}^{\rho} \right\|_{\ell^r}. \tag{29}$$

Similarly,

$$\|v_0\|_{\dot{F}_4^{\sigma,r}} = \frac{Q}{\sqrt{\rho}} \left\| \left\| \left\{ k_s^{\sigma+\frac{3}{2}} |\cos(k_s x_2)| \right\}_{s=1}^{\rho} \right\|_{\ell^r} \right\|_{L^4(\mathbb{T}^2)}, \tag{30}$$

$$\|w_0\|_{\dot{F}_4^{\sigma,r}} = \frac{1}{\sqrt{Q\rho}} \left\| \left\| \left\{ h_s^{\sigma+\frac{3}{2}} |\cos(h_s x_2)| \right\}_{s=1}^{\rho} \right\|_{\ell^r} \right\|_{L^4(\mathbb{T}^2)}. \tag{31}$$

Next we will estimate v_0, w_0, $\mathrm{e}^{t\Delta} v_0$ and $\mathrm{e}^{t\Delta} w_0$.

Lemma 4.1. *Let $r \geqslant 2$ and (v_0, w_0) be given in (27). Then we get*

$$\|v_0\|_{\dot{B}_4^{-\frac{3}{2},r}} + \|\mathrm{e}^{t\Delta} v_0\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}}, \qquad \|w_0\|_{\dot{B}_4^{-\frac{3}{2},2}} + \|\mathrm{e}^{t\Delta} w_0\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim Q^{-\frac{1}{2}}. \tag{32}$$

Proof. From (28)–(29) and $h_s^2 \sim k_s^2$, we get

$$\|v_0\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}}, \qquad \|w_0\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim Q^{-\frac{1}{2}}. \tag{33}$$

Following direct calculation, it is easy to get

$$\mathrm{e}^{t\Delta} v_0 = \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} k_s^{\frac{3}{2}} \mathrm{e}^{-tk_s^2} \cos(k_s x_2), \qquad \mathrm{e}^{t\Delta} w_0 = \frac{1}{\sqrt{Q\rho}} \sum_{s=1}^{\rho} h_s^{\frac{3}{2}} \mathrm{e}^{-th_s^2} \cos(h_s x_2).$$

Noting that $\mathrm{e}^{-th_s^2} \leqslant \mathrm{e}^{-tk_s^2} \leqslant 1$, similar to (33), we can prove the desired estimates for $e^{t\Delta} v_0$ and $\mathrm{e}^{t\Delta} w_0$. □

From the above lemma we observe that for any given $\delta > 0$ and $r > 2$, there exist sufficiently large ρ and Q such that $\|v_0\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim \delta$ and $\|w_0\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim \delta$.

Lemma 4.2. *For any $T \in (0, \infty]$ and let (v_0, w_0) be given in (27), then we have*

$$\|\mathrm{e}^{t\Delta} v_0\|_{L_T^4 \dot{W}^{-1,4} \cap L_T^2 \dot{W}^{-\frac{1}{2},4}} \lesssim \frac{Q}{\sqrt{\rho}} \min\left\{ T^{\frac{1}{4}} |k_\rho|^{\frac{1}{2}} + T^{\frac{1}{2}} |k_\rho|,\ \sqrt{\rho} \right\}, \tag{34}$$

$$\|\mathrm{e}^{t\Delta} w_0\|_{L_T^4 \dot{W}^{-1,4} \cap L_T^2 \dot{W}^{-\frac{1}{2},4}} \lesssim \frac{1}{\sqrt{Q\rho}} \min\left\{ T^{\frac{1}{4}} |k_\rho|^{\frac{1}{2}} + T^{\frac{1}{2}} |k_\rho|,\ \sqrt{\rho} \right\}. \tag{35}$$

Proof. We first estimate the $\|e^{t\Delta}v_0\|_{L^4_T\dot{\mathcal{W}}^{-1,4}(\mathbb{T}^2)}$. Similar to (30), by using the Minkowski inequality we have

$$\begin{aligned}\|e^{t\Delta}v_0\|_{L^4_T\dot{\mathcal{W}}^{-1,4}} &\sim \|e^{t\Delta}v_0\|_{L^4_T\dot{F}_4^{-1,2}}\\ &\lesssim \frac{Q}{\sqrt{\rho}}\Big\|\Big\|\Big(\sum_{s=1}^{\rho}k_s e^{-ctk_s^2}|\cos(k_sx_2)|^2\Big)^{\frac12}\Big\|_{L^4(\mathbb{T}^2)}\Big\|_{L^4_T}\\ &\lesssim \frac{Q}{\sqrt{\rho}}\Big\|\Big(\sum_{s=1}^{\rho}k_s e^{-ctk_s^2}\Big)^{\frac12}\Big\|_{L^4_T} \lesssim \frac{Q}{\sqrt{\rho}}\Big(\sum_{s=1}^{\rho}\Big(1-e^{-cTk_s^2}\Big)\Big)^{\frac14}. \end{aligned}\tag{36}$$

Noting that $1-e^{-cTk_s^2}\lesssim\min\{Tk_s^2,1\}$ and $\sum_{s=1}^{\rho}Tk_s^2\sim Tk_\rho^2$, then (36) is bounded by

$$(36)\lesssim\frac{Q}{\sqrt{\rho}}\min\Big\{T^{\frac14}k_\rho^{\frac12},\ \sqrt{\rho}\Big\}.\tag{37}$$

Next we estimate the $\|e^{t\Delta}v_0\|_{L^2_T\dot{\mathcal{W}}^{-\frac12,4}}$. Similar to (36) and (37), we have

$$\|e^{t\Delta}v_0\|_{L^2_T\dot{\mathcal{W}}^{-\frac12,4}}\lesssim\frac{Q}{\sqrt{\rho}}\Big(\sum_{s=1}^{\rho}(1-e^{-cTk_s^2})\Big)^{\frac12}\lesssim\frac{Q}{\sqrt{\rho}}\min\{T^{\frac12}k_\rho,\ \sqrt{\rho}\}.\tag{38}$$

Combining (37) and (38), we obtain the desired estimates for $e^{t\Delta}v_0$. The estimate for $e^{t\Delta}w_0$ follows in the similar way. Thus we finish the proof. □

Recalling the bilinear estimate for $B(w,v)$ in lemma 2.2 and definitions of y in (24), it is natural to hope that nonlinear terms in y can be well controlled. Therefore, we need to analyse how y evolve in different time scales and see their contributions by using the time-step-division method introduced by Bourgain–Pavlović in [4] to prove norm inflation of v. Let

$$k_\rho^{-2}=T_0<T_1<T_2<\cdots<T_\beta=k_0^{-2},\tag{39}$$

where $\beta=Q^3$, $T_\alpha=k_{\rho_\alpha}^{-2}$, $\rho_\alpha=\rho-\alpha Q^{-3}\rho$ and $\alpha=0,1,2,\ldots,\beta$.

To prove the *a priori* estimates, we recall that $X_{[a,b]}=L^2_{[a,b]}\dot{\mathcal{W}}^{-\frac12,4}\cap L^4_{[a,b]}\dot{\mathcal{W}}^{-1,4}$.

Lemma 4.3. *Suppose that ρ and Q are large enough numbers. Then we have*

$$\|e^{t\Delta}v_0\|_{X_{[T_\alpha,T_{\alpha+1}]}}\lesssim Q^{-\frac12}+\frac{Q}{\sqrt{\rho}},\qquad \|e^{t\Delta}w_0\|_{X_{[T_\alpha,T_{\alpha+1}]}}\lesssim Q^{-2}+\frac{1}{\sqrt{Q\rho}}.\tag{40}$$

Proof. It suffices to estimate $e^{t\Delta}v_0$. Similar to (36), we have

$$\begin{aligned}\|e^{t\Delta}v_0\|_{L^4_{[T_\alpha,T_{\alpha+1}]}\dot{\mathcal{W}}^{-1,4}} &\lesssim\frac{Q}{\sqrt{\rho}}\Big\|\Big(\sum_{s=1}^{\rho}e^{-ctk_s^2}k_s\|\cos(k_sx_2)\|^2_{L^4(\mathbb{T}^2)}\Big)^{\frac12}\Big\|_{L^4_{[T_\alpha,T_{\alpha+1}]}}\\ &\lesssim\frac{Q}{\sqrt{\rho}}\Big(\sum_{s=1}^{\rho}(e^{-cT_\alpha k_s^2}-e^{-cT_{\alpha+1}k_s^2})\Big)^{\frac14}\\ &\lesssim\frac{Q^{\frac14}}{\rho^{\frac14}}+\frac{Q}{\sqrt{\rho}},\end{aligned}\tag{41}$$

where in the last inequality we used

$$\begin{cases} \sum_{s=1}^{\rho_{\alpha+1}-1} (e^{-cT_\alpha k_s^2} - e^{-cT_{\alpha+1}k_s^2}) \lesssim 1, \\ \sum_{s=\rho_{\alpha+1}}^{\rho_\alpha} (e^{-cT_\alpha k_s^2} - e^{-cT_{\alpha+1}k_s^2}) \lesssim \rho Q^{-3}, \\ \sum_{s=\rho_\alpha+1}^{\rho} (e^{-cT_\alpha k_s^2} - e^{-cT_{\alpha+1}k_s^2}) \lesssim 1. \end{cases} \tag{42}$$

Similar to (38), by using (42) we obtain that

$$\|e^{t\Delta}v_0\|_{L^2_{[T_\alpha,T_{\alpha+1}]}\dot{\mathcal{W}}^{-\frac{1}{2},4}} \lesssim \frac{Q}{\sqrt{\rho}}\Big(\sum_{s=1}^{\rho}(e^{-cT_\alpha k_s^2} - e^{-cT_{\alpha+1}k_s^2})\Big)^{\frac{1}{2}} \lesssim Q^{-\frac{1}{2}} + \frac{Q}{\sqrt{\rho}}. \tag{43}$$

Estimates for $e^{t\Delta}w_0$ follow in the similar way. Thus we finish the proof. □

As a direct consequence of (42) with $\rho_\beta = 0$ and

$$\sum_{s=\rho_\beta+1}^{\rho} (e^{-cT_\beta k_s^2} - e^{-cTk_s^2}) = \sum_{s=1}^{\rho}(e^{-cT_\beta k_s^2} - e^{-cTk_s^2}) \lesssim 1,$$

we have the following results.

Lemma 4.4. *Suppose that $T > T_\beta = k_0^{-2}$ ($k_0 = 2^{m_0}$). Then we have*

$$\|e^{t\Delta}v_0\|_{X_{[T_\beta,T]}} \lesssim \frac{Q}{\sqrt{\rho}}, \qquad \|e^{t\Delta}w_0\|_{X_{[T_\beta,T]}} \lesssim \frac{1}{\sqrt{Q\rho}}. \tag{44}$$

4.2. Estimates for the second approximation terms and remainders

Since norm inflation of v comes from part of the bilinear term $v_2 = B(w_1, v_1)$ with $w_1 = e^{t\Delta}w_0$ and $v_1 = e^{t\Delta}v_0$, but w does not have norm inflation property, we only need to split the second approximation term v_2 into three different parts:

- $v_2 = v_{2,0} + v_{2,1} + v_{2,2}$ with

$$\begin{cases} v_{2,0} = \frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\int_0^t e^{-(t-\tau)-\tau(k_s^2+h_s^2)}\, k_s^{\frac{1}{2}}h_s^{\frac{3}{2}}\, d\tau \cos x_2, \\ v_{2,1} = \frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}\int_0^t e^{-(t-\tau)(h_s-k_\ell)^2-\tau(k_\ell^2+h_s^2)}\, k_\ell^{\frac{1}{2}}h_s^{\frac{3}{2}}\,(h_s-k_\ell)\cos((h_s-k_\ell)x_2)\, d\tau, \\ v_{2,2} = \frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell=1}^{\rho}\int_0^t e^{-(t-\tau)(h_s+k_\ell)^2-\tau(h_s^2+k_\ell^2)} h_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}\,(h_s+k_\ell)\cos((h_s+k_\ell)x_2)\, d\tau, \end{cases}$$

- $w_2 = w_{2,1} + w_{2,2}$ with

$$\begin{cases} w_{2,1} = \frac{1}{2Q\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}\int_0^t e^{-(t-\tau)(k_s-k_\ell)^2-\tau(k_\ell^2+k_s^2)}\, k_\ell^{\frac{1}{2}}k_s^{\frac{3}{2}}\,(k_s-k_\ell)\cos((k_s-k_\ell)x_2)\, d\tau, \\ w_{2,2} = \frac{1}{2Q\rho}\sum_{s=1}^{\rho}\sum_{\ell=1}^{\rho}\int_0^t e^{-(t-\tau)(k_s+k_\ell)^2-\tau(k_s^2+k_\ell^2)} k_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}\,(k_s+k_\ell)\cos((k_s+k_\ell)x_2)\, d\tau. \end{cases}$$

Now we prove the lower bound of $v_{2,0}$ which is the most important term that gives us the desired lower bound.

Lemma 4.5. *Let $n=2$, $r\in[1,\infty]$ and $|k_1|^{-2}\ll T\ll 1$. We have*

$$\|v_{2,0}(T)\|_{\dot{B}_4^{-\frac{3}{2},r}}\sim\sqrt{Q},\qquad \|v_{2,0}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}}\lesssim T^{\frac{1}{4}}\sqrt{Q}. \tag{45}$$

Proof. From the definition of v_0 and noting that the frequency of $\cos x_2$ is $\xi=(0,1)$, which is located in an isolated annulus away from zero. From (28), we have

$$\begin{aligned}\|v_{2,0}(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} &\sim \left(\frac{\sqrt{Q}}{\rho}\sum_{s=1}^{\rho}\int_0^T \mathrm{e}^{-(T-\tau)-\tau(k_s^2+h_s^2)}\,k_s^{\frac{1}{2}}h_s^{\frac{3}{2}}\,\mathrm{d}\tau\right)\left\|\cos x_2\right\|_{\dot{B}_4^{-\frac{3}{2},r}(\mathbb{T}^2)}\\ &\sim\left(\frac{\sqrt{Q}}{\rho}\sum_{s=1}^{\rho}\frac{k_s^{\frac{1}{2}}h_s^{\frac{3}{2}}}{k_s^2+h_s^2-1}(\mathrm{e}^{-T}-\mathrm{e}^{-T(k_s^2+h_s^2)})\right)\left\|\cos x_2\right\|_{L^4(\mathbb{T}^2)}\\ &\sim\sqrt{Q}.\end{aligned}$$

From (30) and (31), we have

$$\begin{aligned}\|v_{2,0}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}}&\lesssim\Big\|\frac{\sqrt{Q}}{\rho}\sum_{s=1}^{\rho}\frac{k_s^{\frac{1}{2}}h_s^{\frac{3}{2}}}{k_s^2+h_s^2-1}(\mathrm{e}^{-t}-\mathrm{e}^{-t(k_s^2+h_s^2)})\Big\|_{L_T^4\cap L_T^2}\\ &\lesssim(T^{\frac{1}{4}}+T^{\frac{1}{2}})\sqrt{Q}.\end{aligned}$$

From the assumption of T, we can prove the desired estimates. □

Next we estimate $v_{2,1}$ and $w_{2,1}$.

Lemma 4.6. *Let $n=2$. For any $r\geqslant 2$, we have the following estimates:*

$$\|v_{2,1}\|_{\dot{B}_4^{-\frac{3}{2},r}}+\|v_{2,1}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}}\lesssim\frac{\sqrt{Q}}{\rho}, \tag{46}$$

$$\|w_{2,1}\|_{\dot{B}_4^{-\frac{3}{2},2}}+\|w_{2,1}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}}\lesssim\frac{1}{Q\rho}. \tag{47}$$

Proof. From the definitions of v_1 and w_1 we get

$$v_{2,1}=\frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}h_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}(h_s-k_\ell)\frac{\mathrm{e}^{-t(h_s-k_\ell)^2}-\mathrm{e}^{-t(h_s^2+k_\ell^2)}}{h_s^2+k_\ell^2-(h_s-k_\ell)^2}\cos((h_s-k_\ell)x_2), \tag{48}$$

$$w_{2,1}=\frac{1}{2Q\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}k_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}(k_s-k_\ell)\frac{\mathrm{e}^{-t(k_s-k_\ell)^2}-\mathrm{e}^{-t(k_s^2+k_\ell^2)}}{k_s^2+k_\ell^2-(k_s-k_\ell)^2}\cos((k_s-k_\ell)x_2). \tag{49}$$

By making use of the lacunary properties of $\{h_s\}_{s=1}^{\rho}$, $\{k_\ell\}_{\ell=1}^{\rho}$ and (28), we have

$$\begin{aligned}\|v_{2,1}\|_{\dot{B}_4^{-\frac{3}{2},r}}&\lesssim\|v_{2,1}\|_{\dot{B}_4^{-\frac{3}{2},1}}\\ &\lesssim\frac{\sqrt{Q}}{\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}\frac{h_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}|h_s-k_\ell|}{(h_s^2+k_\ell^2-(h_s-k_\ell)^2)|h_s-k_\ell|^{\frac{3}{2}}}\|\cos((h_s-k_\ell)x_2)\|_{L^4(\mathbb{T}^2)}\\ &\lesssim\frac{\sqrt{Q}}{\rho}\Big(\sum_{s=1}^{\rho}\sum_{\ell<s}\frac{k_\ell^{\frac{1}{2}}}{k_s}+\sum_{s=1}^{\rho}\sum_{\ell>s}\frac{k_s^{\frac{3}{2}}}{k_\ell^2}\Big)\\ &\lesssim\frac{\sqrt{Q}}{\rho}.\end{aligned}$$

By making use of (30), (48) and $|h_s - k_\ell| = \max\{h_s, k_\ell\}$, we obtain that

$$\|\mathrm{e}^{-t(h_s-k_\ell)^2} - \mathrm{e}^{-t(h_s^2+k_\ell^2)}\|_{L_T^4} \lesssim \frac{1}{|h_s-k_\ell|^{\frac{1}{2}}}, \qquad \|\mathrm{e}^{-t(h_s-k_\ell)^2} - \mathrm{e}^{-t(h_s^2+k_\ell^2)}\|_{L_T^4} \lesssim \frac{1}{|h_s-k_\ell|}.$$

Hence we get

$$\begin{aligned}\|v_{2,1}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}} &\lesssim \frac{\sqrt{Q}}{\rho}\Big(\sum_{s=1}^{\rho}\sum_{\ell\neq s}\frac{h_s^3k_\ell}{|h_s-k_\ell|^5}\|\cos((h_s-k_\ell)x_2)\|_{L^4(\mathbb{T}^2)}^2\Big)^{\frac{1}{2}}\\ &\lesssim \frac{\sqrt{Q}}{\rho}\left(\sum_{s=1}^{\rho}\sum_{\ell<s}\frac{k_\ell}{k_s^2} + \sum_{s=1}^{\rho}\sum_{\ell>s}\frac{k_s^3}{k_\ell^4}\right)^{\frac{1}{2}}\\ &\lesssim \frac{\sqrt{Q}}{\rho}.\end{aligned}$$

Estimates for $w_{2,1}$ follow in the similar way. Hence we omit the details. □

It remains to estimate $v_{2,2}$ and $w_{2,2}$. By checking the proof of Lemma 4.6, we observe that the following results are also true.

Lemma 4.7. *Let $n = 2$. For any $r \geqslant 2$, we have*

$$\|v_{2,2}\|_{\dot{B}_4^{-\frac{3}{2},r}} + \|v_{2,2}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}} \lesssim \frac{\sqrt{Q}}{\rho}, \tag{50}$$

$$\|w_{2,2}\|_{\dot{B}_4^{-\frac{3}{2},2}} + \|w_{2,2}\|_{L_T^4\dot{\mathcal{W}}^{-1,4}\cap L_T^2\dot{\mathcal{W}}^{-\frac{1}{2},4}} \lesssim \frac{1}{Q\rho}. \tag{51}$$

We are now ready to estimate the remainder y and z. Recall that y and z satisfy the following integral equations (see (24)–(26)):

$$\begin{cases} y = B(z, y + v_1 + v_2) + B(w_1+w_2, y) + B(w_1, v_2) + B(w_2, v_1 + v_2),\\ z = B(y, y + v_1 + v_2) + B(v_1 + v_2, y) + B(v_1, v_2) + B(v_2, v_1 + v_2), \end{cases} \tag{52}$$

on $(0, \infty)$ with the initial conditions $(y(0), z(0)) = (0, 0)$.

To explore the delicate decay estimates of y and z, we split each of v_1, w_1, v_2, w_2, y and z into another two terms, i.e.

$$\begin{cases} v_1 = v_1\chi_{[0,T_\sigma]}(t) + v_1\chi_{[T_\sigma,T_{\sigma+1}]}(t),\\ v_2 = v_2\chi_{[0,T_\sigma]}(t) + v_2\chi_{[T_\sigma,T_{\sigma+1}]}(t),\\ y = y\chi_{[0,T_\sigma]}(t) + y\chi_{[T_\sigma,T_{\sigma+1}]}(t), \end{cases} \qquad \begin{cases} w_1 = w_1\chi_{[0,T_\sigma]}(t) + w_1\chi_{[T_\sigma,T_{\sigma+1}]}(t),\\ w_2 = w_2\chi_{[0,T_\sigma]}(t) + w_2\chi_{[T_\sigma,T_{\sigma+1}]}(t),\\ z = z\chi_{[0,T_\sigma]}(t) + Z\chi_{[T_\sigma,T_{\sigma+1}]}(t). \end{cases}$$

Applying lemma 2.2 to y and z, and considering the time decomposition, we have the following iterated results.

Lemma 4.8. *Suppose that y and z satisfy (24)–(26). For any $\alpha \in \{0, 1, \cdots, \beta\}$ with $\beta = Q^3$ and $T_\beta = k_0^{-2}$, and for sufficiently large ρ and k_0, we have*

$$\|y\|_{X_{T_\beta}} + \|z\|_{X_{T_\beta}} \lesssim Q^{Q^3+1}\left(\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{k_0}}\right). \tag{53}$$

Moreover, for any $T > T_\beta = k_0^{-2}$, we have

$$\|y\|_{X_T} + \|z\|_{X_T} \lesssim Q^{\frac{3}{2}}(\frac{1}{\rho} + T^{\frac{1}{4}}) + Q^{Q^3+2}\left(\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{k_0}}\right). \tag{54}$$

Proof. Noting that $[0, T_{\alpha+1}] = [0, T_\alpha] \cup [T_\alpha, T_{\alpha+1}]$, we thus have

$$\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}} \lesssim (\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}) + (\|y\|_{X_{[T_\alpha,T_{\alpha+1}]}} + \|z\|_{X_{[T_\alpha,T_{\alpha+1}]}}). \tag{55}$$

For simplicity, here and hereafter, we denote

$$\begin{cases} A_\alpha := \|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}, \\ A_{\alpha,\alpha+1} := \|y\|_{X_{[T_\alpha,T_{\alpha+1}]}} + \|z\|_{X_{[T_\alpha,T_{\alpha+1}]}}, \\ B_\alpha := \|v_1\|_{X_{T_\alpha}} + \|w_1\|_{X_{T_\alpha}}, \\ B_{\alpha,\alpha+1} := \|v_1\|_{X_{[T_\alpha,T_{\alpha+1}]}} + \|w_1\|_{X_{[T_\alpha,T_{\alpha+1}]}}, \\ C_\alpha := \|v_2\|_{X_{T_\alpha}} + \|w_2\|_{X_{T_\alpha}}, \\ C_{\alpha,\alpha+1} := \|v_2\|_{X_{[T_\alpha,T_{\alpha+1}]}} + \|w_2\|_{X_{[T_\alpha,T_{\alpha+1}]}}. \end{cases}$$

Applying lemma 2.4 and lemmas 4.1–4.7 to (52), we have

$$\begin{aligned} A_\alpha &\lesssim A_\alpha^2 + A_\alpha(B_\alpha + C_\alpha) + C_\alpha(B_\alpha + C_\alpha) \\ &\lesssim A_\alpha^2 + A_\alpha(Q + Q^{\frac12}\rho^{-1} + T_\alpha^{\frac14}Q^{\frac12}) + (Q^{\frac12}\rho^{-1} + T_\alpha^{\frac14}Q^{\frac12})(Q + Q^{\frac12}\rho^{-1} + T_\alpha^{\frac14}Q^{\frac12}) \\ &\lesssim A_\alpha^2 + QA_\alpha + Q^{\frac32}(\rho^{-1} + T_\beta^{\frac14}). \end{aligned} \tag{56}$$

Similarly, suppose that $T_\beta \ll Q^{-4}$ and $\rho \gg Q^3$ (or $T = o(Q^{-4})$ and $\rho^{-1} = o(Q^{-3})$). Then by applying lemma 2.4 and lemmas 4.1–4.7 to (52), we have

$$\begin{aligned} A_{\alpha,\alpha+1} &\lesssim A_{\alpha,\alpha+1}^2 + A_{\alpha,\alpha+1}(B_{\alpha,\alpha+1} + C_{\alpha,\alpha+1}) + C_{\alpha,\alpha+1}(B_{\alpha,\alpha+1} + C_{\alpha,\alpha+1}) \\ &\quad + A_\alpha^2 + A_\alpha(B_\alpha + C_\alpha) + C_\alpha(B_\alpha + C_\alpha) \\ &\lesssim A_{\alpha+1}^2 + A_{\alpha,\alpha+1}(Q^{-\frac12} + Q\rho^{-\frac12} + T_{\alpha+1}^{\frac14}Q^{\frac12}) + QA_\alpha \\ &\quad + (Q^{\frac12}\rho^{-1} + T_{\alpha+1}^{\frac14}Q^{\frac12})(Q + Q^{-\frac12} + Q\rho^{-\frac12} + T_{\alpha+1}^{\frac14}Q^{\frac12}) \\ &\lesssim Q^{\frac32}(\rho^{-1} + T_\beta^{\frac14}) + QA_\alpha + o(1)A_{\alpha,\alpha+1} + A_{\alpha+1}^2. \end{aligned} \tag{57}$$

Plugging (56) and (57) into (55), we obtain that if $A_\beta \lesssim o(1)$, then

$$A_{\alpha+1} \lesssim A_\alpha + A_{\alpha,\alpha+1} \lesssim Q^{\frac32}(\rho^{-1} + T_\beta^{\frac14}) + QA_\alpha + o(1)A_{\alpha+1}. \tag{58}$$

Noting that from (39), (34) and (35), $A_0 \lesssim Q\rho^{-\frac12}$. Thus by iterating (58) we get

$$A_\beta \lesssim Q^{\beta+1}(\rho^{-\frac12} + T_\beta^{\frac14}). \tag{59}$$

For sufficiently large ρ and T_β^{-1}, we obtain that $A_\beta \lesssim o(1)$ and the above iterating argument works. Therefore, we prove (53).

In order to prove (54), it suffices to iterate one more time by splitting $[0, T]$ into $[0, T_\beta] \cup [T_\beta, T]$. Similar to (58), by using (59) and lemma 4.4, we have

$$\begin{aligned} \|y\|_{X_T} + \|z\|_{X_T} &\lesssim Q^{\frac32}(\rho^{-1} + T^{\frac14}) + QA_\beta \\ &\lesssim Q^{\frac32}(\rho^{-1} + T^{\frac14}) + Q^{\beta+2}(\rho^{-\frac12} + T_\beta^{\frac14}). \end{aligned}$$

Recalling that $\beta = Q^3$ and $T_\beta = k_0^{-2}$, then we prove (54). □

It is clear that from lemma 2.4 and (58), we can prove that

$$\|y(T)\|_{\dot B_4^{-\frac32,2}} + \|z(T)\|_{\dot B_4^{-\frac32,2}} \lesssim Q^{\frac32}(\rho^{-1} + T^{\frac14}) + Q^{\beta+3}(\rho^{-\frac12} + T_\beta^{\frac14}). \tag{60}$$

4.3. Proof of theorem 1.4

Gathering the above estimates, we are ready to prove the ill-posedness of system (1) by showing norm inflation.

Proof of theorem 1.4. Combining lemma 2.4, (32), (45), (46) and (54), we have

$$\begin{aligned}
\|v(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} &\geqslant \|v_{2,0}(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} - (\|v_1(T)+v_{2,1}(T)+y(T)\|_{\dot{B}_4^{-\frac{3}{2},r}}) \\
&\geqslant \|v_{2,0}(T)\|_{L^4} - \|v_1(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} - \|v_{2,1}(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} - \|y(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} \\
&\geqslant c\,Q^{\frac{1}{2}} - C\Big(Q\rho^{-\frac{1}{2}+\frac{1}{r}} + Q^{\frac{1}{2}}\rho^{-1} + Q^{\frac{3}{2}}(\rho^{-1}+T^{\frac{1}{4}}) + Q^{\beta+3}(\rho^{-\frac{1}{2}}+T_\beta^{\frac{1}{4}})\Big) \\
&\geqslant \frac{1}{2}c\,Q^{\frac{1}{2}}
\end{aligned} \tag{61}$$

provided that $\rho \geqslant \max\{Q^{\frac{r}{r-2}}, Q^{2Q^3+7}\}$, $k_0 \gg Q^{2Q^3+7}$ and $T \ll Q^{-12}$. However, w does not produce norm inflation since

$$\begin{aligned}
\|w(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} &\lesssim \|w_1(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} + \|w_2(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} + \|z(T)\|_{\dot{B}_4^{-\frac{3}{2},2}}) \\
&\lesssim Q^{-\frac{1}{2}} + Q^{-1}\rho^{-1} + Q^{\frac{3}{2}}(\rho^{-1}+T^{\frac{1}{4}}) + Q^{\beta+3}(\rho^{-\frac{1}{2}}+T_\beta^{\frac{1}{4}})\Big) \\
&\lesssim Q^{-\frac{1}{2}}
\end{aligned}$$

when $\rho \geqslant \max\{Q^{\frac{3r}{r-2}}, Q^{2Q^3+7}\}$, $k_0 \gg Q^{2Q^3+7}$ and $T \ll Q^{-12}$. □

4.4. Outline of the proof of theorem 1.5

In order to effectively communicate the ideas in the proof, this subsection outlines the main steps. Due to similarity, further details on the intermediate steps are omitted for simplicity.

Step 1. Fix a real number $\delta > 0$.

$$v_0(x) = \frac{1}{\sqrt{Q\rho}}\sum_{s=1}^{\rho} k_s^{\frac{3}{2}}\cos(k_s x_2), \qquad w_0(x) = \frac{Q}{\sqrt{\rho}}\sum_{s=1}^{\rho} h_s^{\frac{3}{2}}\cos(h_s x_2), \tag{62}$$

where $x \in \mathbb{T}^2$, $k_s = 2^{\frac{(s+1)(s+2m_0)}{2}}$, $h_s = 1 + k_s$ and m_0 is a large number depending on δ. Split v_2 and w_2 from (23) into $v_2 = v_{2,0} + v_{2,1} + v_{2,2}$ and $w_2 = w_{2,0} + w_{2,1}$, respectively, where

$$v_{2,0} = \frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\int_0^t \mathrm{e}^{-(t-\tau)-\tau(k_s^2+h_s^2)}\,k_s^{\frac{1}{2}}h_s^{\frac{3}{2}}\,\mathrm{d}\tau\,\cos x_2,$$

$$v_{2,1} = \frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}\int_0^t \mathrm{e}^{-(t-\tau)(h_s-k_\ell)^2-\tau(k_\ell^2+h_s^2)}\,k_\ell^{\frac{1}{2}}h_s^{\frac{3}{2}}(h_s-k_\ell)\cos((h_s-k_\ell)x_2)\,\mathrm{d}\tau,$$

$$v_{2,2} = \frac{\sqrt{Q}}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell=1}^{\rho}\int_0^t \mathrm{e}^{-(t-\tau)(h_s+k_\ell)^2-\tau(h_s^2+k_\ell^2)}h_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}(h_s+k_\ell)\cos((h_s+k_\ell)x_2)\mathrm{d}\tau,$$

$$w_{2,1} = \frac{Q^2}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell\neq s}\int_0^t \mathrm{e}^{-(t-\tau)(k_s-k_\ell)^2-\tau(k_\ell^2+k_s^2)}\,k_\ell^{\frac{1}{2}}k_s^{\frac{3}{2}}(k_s-k_\ell)\cos((k_s-k_\ell)x_2)\,\mathrm{d}\tau,$$

$$w_{2,2} = \frac{Q^2}{2\rho}\sum_{s=1}^{\rho}\sum_{\ell=1}^{\rho}\int_0^t \mathrm{e}^{-(t-\tau)(k_s+k_\ell)^2-\tau(k_s^2+k_\ell^2)}k_s^{\frac{3}{2}}k_\ell^{\frac{1}{2}}(k_s+k_\ell)\cos((k_s+k_\ell)x_2)\,\mathrm{d}\tau,$$

and $v_{2,0}$ exhibits norm inflation while $v_{2,1}$, $w_{2,1}$, $v_{2,2}$ and $w_{2,2}$ are controllable terms.

Step 2. With our careful choice of initial conditions and making appropriate choices for Q, ρ, k_0, and T, following the similar arguments as in section 4.2, we can prove that

- $\|v_1(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} \lesssim Q^{-\frac{1}{2}}$, $\|w_1(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} \lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}}$,
- $\|v_{2,0}(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} \sim Q^{\frac{1}{2}}$,
- $\|v_{2,1}(T)+v_{2,2}(T)\|_{\dot{B}_2^{-\frac{3}{2},2}} \lesssim \frac{Q^{\frac{1}{2}}}{\rho}$, $\|w_{2,1}(T)+w_{2,2}(T)\|_{\dot{B}_2^{-\frac{3}{r},2}} \lesssim \frac{Q^2}{\rho}$,
- $\|y\|_{X_T}+\|z\|_{X_T} \lesssim Q^3(\frac{1}{\rho}+T^{\frac{1}{4}})+Q^{Q^3+N_0}(\frac{1}{\sqrt{\rho}}+\frac{1}{\sqrt{k_0}})$ for some $N_0 \gg 1$.

Step 3. (Norm inflation). The estimates in step 2 and similar estimate (60) imply

$$\begin{aligned}\|v(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} &\geqslant \|v_{2,0}(T)\|_{\dot{B}_4^{-\frac{3}{2},2}} - \Big\||v_1(T)|+|v_{2,1}(T)+v_{2,2}(T)|+|y(T)|\Big\|_{\dot{B}_4^{-\frac{3}{2},2}}\\ &\gtrsim 1/\delta\end{aligned}$$

and there exists some $N_0 \in \mathbb{N}$ such that

$$\begin{aligned}\|w(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} &\lesssim \|w_1(T)\|_{\dot{B}_4^{-\frac{3}{2},r}} + \Big\||w_1(T)|+|w_{2,1}(T)+u_{2,2}(T)|+|y(T)|\Big\|_{\dot{B}_4^{-\frac{3}{2},2}}\\ &\lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}}+Q^3(\rho^{-1}+T^{\frac{1}{4}})+Q^{Q^3+N_0+1}(\rho^{-\frac{1}{2}}+k_0^{-\frac{1}{2}})\\ &\lesssim \delta\end{aligned}$$

provided that ρ and k_0 are large enough and $T(>k_0^2)$ is small enough. □

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References

[1] Arnold A, Markowich P, Toscani G and Unterreiter A 2001 On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations *Commun. Part. Diff. Eqns* **26** 43–100
[2] Biler P and Dolbeault J 2000 Long time behavior of solutions to Nernst–Plank and Debye–Hückel drift–diffusion systems *Ann. Hénri Poincaré* **1** 461–72
[3] Biler P, Hebisch W and Nadzieja T 1994 The Debye system: existence and large time behavior of solutions *Nonlinear Anal. TMA* **23** 1189–209
[4] Bourgain J and Pavlović N 2008 Ill-posedness of the Navier–Stokes equations in a critical space in 3D *J. Funct. Anal.* **255** 2233–47
[5] Chae D 2002 On the well-posedness of the Euler equations in the Triebel–Lizorkin spaces *Commun. Pure Appl. Math.* **55** 654–78
[6] Debye P and Hückel E 1923 De la theorie des electrolytes. I. Abaissement du point de congelation et phenomenes associes *Phys. Zft.* **24** 185–206
[7] Chen H and Zhong X 2004 Norm behavior of solutions to a parabolic–elliptic system modelling chemotaxis in a domain of R^3 *Math. Methods Appl. Sci.* **27** 991–1006

[8] Deng C and Liu C Largest well-posed spaces for the general diffusion equations with nonlocal interactions, submitted

[9] Deng C and Yao X 2014 Well-posednesss and ill-posedness of the 3D generalized Navier–Stokes equations in $\dot{F}^{-\alpha,r}_{\frac{3}{\alpha-1}}$ *Discrete Cont. Dyn. Sys.* **34** 437–59

[10] Fan J and Gao H 2009 Uniqueness of weak solutions to a non-linear hyperbolic system in electrohydrodynamics *Nonlinear Anal.* **70** 2382–6

[11] Fefferman C and Stein E 1971 Some maximal inequalities *Am. J. Math.* **93** 107–15

[12] Fujita H and Kato T 1964 On the Navier–Stokes initial value problem: I *Arch. Ration. Mech. Anal.* **16** 269–315

[13] Gajewski H 1985 On existence, uniqueness and asymptotic behavior of solutions of the basic equations for carrier transport in semiconductors *Z. Angew. Math. Mech.* **65** 101–8

[14] Gajewski H and Gröger K 1986 On the basic equations for carrier transport in semiconductors *J. Math. Anal. Appl.* **113** 12–35

[15] Horstmann D 2003 From 1970 until present: the Keller–Segel model in chemotaxis and its consequences *Jahresber. Deutsch. Math. Ver.* **105** 103–65

[16] Iwabuchi T 2011 Global well-posedness for Keller–Segel system in Besov type spaces *J. Math. Anal. Appl.* **379** 930–48

[17] Kurokiba M and Ogawa T 2008 Well-posedness for the drift–diffusion system in L^p arising from the semiconductor device simulation *J. Math. Anal. Appl.* **342** 1052–67

[18] Lemarié-Rieusset P G 2010 *Recent Developments in the Navier–Stokes Problems* (Boca Raton, FL: CRC)

[19] Li F 2009 Quasineutral limit of the electro-diffusion model arising in electrohydrodynamics *J. Diff. Eqns* **246** 3620–41

[20] Liu W 2009 One-dimensional steady-state Poisson–Nernst–Planck systems for ion channels with multiple ion species *J. Diff. Eqns* **246** 428–51

[21] Liu W and Wang B 2010 Poisson–Nernst–Planck systems for narrow tubular-like membrane channels *J. Dyn. Diff. Eqns* **22** 413–37

[22] Mock M 1972 On equations describing steady-state carrier distributions in a semiconductor device *Commun. Pure Appl. Math.* **25** 781–92

[23] Mock M 1974 An initial value problem from semiconductor device theory *SIAM J. Math. Anal.* **5** 597–612

[24] Ogawa T and Shimizu S 2008 The drift–diffusion system in two-dimensional critical hardy space *J. Funct. Anal.* **255** 1107–38

[25] Ogawa T and Shimizu S 2010 End-point maximal regularity and its application to two-dimensional Keller–Segel system *Math. Z.* **264** 601–28

[26] Roubicek T 2006 Incompressible ionized fluid mixtures *Contin. Mech. Thermodyn.* **17** 493–509

[27] Roubicek T 2007 Incompressible ionized non-Newtonian fluid mixtures *SIAM J. Math. Anal.* **39** 863–90

[28] Ryham R, Liu C and Wang Z 2006 On electro-kinetic fuids: one dimensional configurations *Discrete Contin. Dyn. Syst.* B **6** 357–71

[29] Ryham R, Liu C and Zikatanov L 2007 Mathematical models for the deformation of electrolyte droplets *Discrete Contin. Dyn. Syst.* B **8** 649–61

[30] Selberherr S 1984 *Analysis and Simulation of Semiconductor Devices* (Vienna: Springer)

[31] Triebel H 1995 *Interpolation Theory, Function Spaces, Differential Operators* vol 1 (Heidelberg: Johann Ambrosius Barth)

[32] Wang S and Wang K 2012 The mixed layer problem and quasi-neutral limit of the drift–diffusion model for semiconductors *SIAM J. Math. Anal.* **44** 699–717

[33] Wolansky G 2002 Multi-components chemotactic system in the absence of conflicts *Eur. J. Appl. Math.* **13** 641–61

[34] Zhao J, Liu Q and Cui S 2013 Existence of solutions for the Debye-Hückel system with low regularity initial data *Acta Appl. Math.* **125** 1–10