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Whitney-Graustein homotopy of locally convex curves via a curvature flow. (English. English summary)

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Summary: "Let X_0, $\tilde{X}$ be two smooth, closed and locally convex curves in the plane with same winding number. A curvature flow with a nonlocal term is constructed to evolve X_0 into $\tilde{X}$. It is proved that this flow exits globally, preserves both the local convexity and the elastic energy of the evolving curve. If the two curves have same elastic energy then the curvature flow deforms the evolving curve into the target curve $\tilde{X}$ as time tends to infinity."

The result fits into a larger picture, beginning with the Whitney-Graustein theorem stating that any two smooth, closed curves may be smoothly deformed into each other if and only if they have the same winding number [H. Whitney, Compositio Math. **4** (1937), 276–284; MR1556973]. A famous question of Yau asks how to construct a parabolic curvature flow that realises the Whitney-Graustein deformation. An important result in this direction is that of Lin-Tsai, defining such a parabolic flow that answers Yau's question (up to scale) provided the flow exists for all time [Y.-C. Lin and D.-H. Tsai, J. Differential Equations **247** (2009), no. 9, 2620–2636; MR2568066]. A number of other results have been obtained in this direction, with Gao and Zhang producing a nonlocal flow that fully answers the question in the case of convex curves; that is, smooth boundaries of convex bodies [L. Gao and Y. T. Zhang, J. Differential Equations **266** (2019), no. 1, 179–201; MR3870561].

The contribution here is to introduce a flow that answers Yau's question in the case of smooth, closed, locally convex curves, namely those with positive curvature, but not necessarily simple. The answer is affirmative among such curves with the same winding number, up to scale; more precisely, among such curves with the same elastic energy $\int \kappa^2 ds$.

The (somewhat complicated) flow is nonlocal and fully nonlinear. It is phrased in terms of the support function and radial function. Short-time existence and uniqueness is proven, and a Harnack inequality is derived which is then used to bootstrap higher order estimates. Convergence is obtained first by compactness and then by showing that the limits are unique due to the preservation of elastic energy.

To finish, a detailed example is presented with supporting numerics and figures to illustrate the result. *Paul Bryan*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.