MR3870561 53C44 35K15 53A04

Gao, Laiyuan (PRC-XUNU-SMS); **Zhang, Yuntao** (PRC-XUNU-SMS)

On Yau's problem of evolving one curve to another: convex case. (English. English summary)

J. Differential Equations **266** (2019), *no.* 1, 179–201.

The paper under review deals with a problem of S. T. Yau's that asks whether one can use a parabolic curvature flow method to evolve one planar curve to another.

Extending previous results by Y.-C. Lin and D.-H. Tsai and by K.-S. Chou and X.-P. Zhu, the authors introduce a novel curvature flow and tackle the problem for the case of convex curves.

Namely, let two smooth, closed, convex curves $X_0\colon S^1 \to \mathbb{R}^2$ and $\tilde{X}\colon S^1 \to \mathbb{R}^2$ be prescribed. Consider a one-parameter family of smooth closed curves $X\colon S^1 \times [0,\omega) \to \mathbb{R}^2$ whose evolution in time t is governed by the equation

$$\frac{\partial X}{\partial t}(\varphi,t) = \left(k(\varphi,t) - \lambda(t)\tilde{k}(\varphi)\right) N(\varphi,t), \quad (\varphi,t) \in S^1 \times [0,\omega),$$

$$X(\varphi,0) = X_0(\varphi), \quad \varphi \in S^1,$$

where $X(\varphi,t)$ describes the evolving curve, $k(\varphi,t)$ and $N(\varphi,t)$ are its curvature and inner unit normal respectively, and $\tilde{k}(\varphi)$ stands for the curvature of the target curve $\tilde{X}$. The nonlocal term λ is equal to $\int_{S^1} \tilde{k} g d\varphi$, where g is the metric of X; it is chosen so that the area enclosed by X remains constant during the evolution.

The main result of the paper states that the flow exists for all $t \in [0,+\infty)$, preserves the convexity, and deforms the initial curve X_0 into a curve X_∞, which is congruent to a homothetic copy of $\tilde{X}$, as $t \to +\infty$. *Vasyl Gorkavyy*

[References]

1. U. Abresch, J. Langer, The normalized curve shortening flow and homothetic solutions, J. Differential Geom. 23 (2) (1986) 175–196. MR0845704
2. B. Andrews, Volume-preserving anisotropic mean curvature flow, Indiana Univ. Math. J. 50 (2) (2001) 783–827. MR1871390
3. S.B. Angenent, On the formation of singularities in the curve shortening flow, J. Differential Geom. 33 (3) (1991) 601–633. MR1100205
4. S.B. Angenent, M.E. Gurtin, Multiphase thermomechanics with interfacial structure 2. Evolution of an isothermal interface, Arch. Ration. Mech. Anal. 108 (1989) 323–391. MR1013461
5. S.B. Angenent, M.E. Gurtin, Anisotropic motion of a phase interface, J. Reine Angew. Math. 446 (1994) 1–47. MR1256146
6. K.-S. Chou, K.-S. Tso, Deforming a hypersurface by its Gauss–Kronecker curvature, Comm. Pure Appl. Math. 38 (1985) 867–882. MR0812353
7. K.-S. Chou, X.-P. Zhu, Anisotropic flows for convex plane curves, Duke Math. J. 97 (3) (1999) 579–619. MR1682990
8. K.-S. Chou, X.-P. Zhu, A convexity theorem for a class of anisotropic flows of plane curves, Indiana Univ. Math. J. 48 (1) (1999) 139–154. MR1722196
9. K.-S. Chou, X.-P. Zhu, The Curve Shortening Problem, CRC Press, Boca Raton, FL, 2001. MR1888641
10. M.E. Gage, An isoperimetric inequality with applications to curve shortening, Duke Math. J. 509 (4) (1983) 1225–1229. MR0726325
11. M.E. Gage, On an area-preserving evolution equation for plane curves, in: D.M. DeTurck (Ed.), Nonlinear Problems in Geometry, in: Contemp. Math., vol. 51, 1986, pp. 51–62. MR0848933

12. M.E. Gage, Positive centers and the Bonnesen inequality, Proc. Amer. Math. Soc. 110 (4) (1990) 1041–1048. MR1042266
13. M.E. Gage, Evolving plane curves by curvature in relative geometries, Duke Math. J. 72 (2) (1993) 441–466. MR1248680
14. M.E. Gage, R.S. Hamilton, The heat equation shrinking convex plane curves, J. Differential Geom. 23 (1) (1986) 69–96. MR0840401
15. M.E. Gage, Evolving plane curves by curvature in relative geometries. II, Duke Math. J. 75 (1) (1994) 79–98. MR1284816
16. M. Green, S. Osher, Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves, Asian J. Math. 3 (1999) 659–676. MR1793675
17. M. Gurtin, Thermomechanics of Evolving Phase Boundaries in the Plane, Oxford Math. Monographs, Oxford University Press, 1993. MR1402243
18. R.S. Hamilton, Four-manifolds with positive curvature operator, J. Differential Geom. 24 (1986) 153–179. MR0862046
19. L.-S. Jiang, S.-L. Pan, On a non-local curve evolution problem in the plane, Comm. Anal. Geom. 16 (2008) 1–26. MR2411467
20. Y.-C. Lin, D.-H. Tsai, On a simple maximum principle technique applied to equations on the circle, J. Differential Equations 245 (2008) 377–391. MR2428003
21. Y.-C. Lin, D.-H. Tsai, Evolving a convex closed curve to another one via a length-preserving linear flow, J. Differential Equations 247 (2009) 2620–2636. MR2568066
22. Y.-C. Lin, D.-H. Tsai, Remarks on some isoperimetric properties of the $k-1$ flow, Pacific J. Math. 260 (1) (2012) 197–213. MR3001791
23. R. Osserman, Bonnesen-style isoperimetric inequalities, Amer. Math. Monthly 86 (1979) 1–29. MR0519520

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.