MR4629880 53E99 35K05 53A04

Gao, Laiyuan (PRC-XUNU-SMS); **Pan, Shengliang** (PRC-TONG-SM)

Star-shaped centrosymmetric curves under Gage's area-preserving flow. (English. English summary)

J. Geom. Anal. **33** (2023), *no.* 11, *Paper No.* 348, 25 *pp.*

The paper under review concerns the theory of geometric evolution flows for closed planar curves. Particularly, the Gage area preserving flow is considered so that the evolution of a planar closed curve $X\colon S^1\times[0,T)\to\mathbb{E}^2$ is governed by the equation

$$\begin{cases} \dfrac{\partial X}{\partial t}(\varphi,t)=\left(k(\varphi,t)-\dfrac{2\pi}{L(t)}\right)N(\varphi,t),\\ X(\varphi,0)=X_0(\varphi),\end{cases}$$

where k, L and N stand for the curvature, the length and the unit inward normal of the evolving curve, respectively [see M. E. Gage, in *Nonlinear problems in geometry (Mobile, Ala., 1985)*, 51–62, Contemp. Math., 51, Amer. Math. Soc., Providence, RI, 1986; MR0848933].

If the initial curve $X_0\colon S^1\to\mathbb{E}^2$ is assumed to be *convex*, then the flow exists for all $t\in[0,+\infty)$ so that the evolving curve converges to a circle. On the other hand, there are non-convex examples where the discussed convergence does not hold. Thus, the problem is to find natural classes of closed planar curves where the convergence of the Gage area preserving flow holds true.

The authors discuss the case of embedded *star-shaped* closed curves in $\mathbb{E}^2$ and prove the following statement as the main result.

Theorem. Let X_0 be a smooth embedded star-shaped closed curve in $\mathbb{E}^2$. If X_0 is symmetric with respect to its star center O, then the Gage area preserving flow exists for all $t\in[0,+\infty)$ so that the evolving curve remains smooth, becomes convex in finite time and converges to a circle centered at O as $t\to+\infty$.

The question of whether the proved statement holds true without the extra symmetry requirement seems to be open. Actually, if X_0 is not assumed to be centrosymmetric, then the evolving curve might lose its star-shapedness, which would cause extra difficulties in studying its asymptotic behavior. *Vasyl Gorkavyy*

[References]

1. Chao, X.-L., Ling, X.-R., Wang, X.L.: On a planar area-preserving curvature flow. Proc. Amer. Math. Soc. **141**(5), 1783–1789 (2013) MR3020863
2. Chou, K.-S., Zhu, X.-P.: The curve shortening problem. CRC Press, Boca Raton (2001) MR1888641
3. Gage, M.E.: An isoperimetric inequality with applications to curve shortening. Duke Math. J. **50**(4), 1225–1229 (1983) MR0726325
4. Gage, M.E.: Curve shortening makes convex curves circular. Invent. Math. **76**(2), 357–364 (1984) MR0742856
5. Gage, M.: On an area-preserving evolution equation for plane curves. Nonlinear Problems Geom. **51**, 51–62 (1986) MR0848933
6. Gage, M.E.: Evolving plane curves by curvature in relative geometries. Duke Math. J. **72**(2), 441–466 (1993) MR1248680
7. Gage, M.E., Hamilton, R.S.: The heat equation shrinking convex plane curves. J. Differ. Geom. **23**(1), 69–96 (1986) MR0840401
8. Gage, M.E., Li, Y.: Evolving plane curves by curvature in relative geometries. II. Duke Math. J. **75**(1), 79–98 (1994) MR1284816
9. Gao, L.-Y., Zhang, Y.-T.: On Yau's problem of evolving one curve to another:

convex case. J. Differ. Equ. **266**(1), 179–201 (2019) MR3870561
10. Grayson, M.A.: The heat equation shrinks embedded plane curve to round points. J. Differ. Geom. **26**(2), 285–314 (1987) MR0906392
11. Grayson, M.A.: Shortening embedded curves. Ann. Math. **129**(1), 71–111 (1989) MR0979601
12. Gruber, P.M.: Convex and discrete geometry. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **336**. Springer, Berlin, (2007) MR2335496
13. Hsiung, C.-C.: A first course in differential geometry. Pure & Applied Math. Wiley, New York (1981) MR0608028
14. Huisken, G.: The volume preserving mean curvature flow. J. Reine Angew. Math. **382**, 35–48 (1987) MR0921165
15. Huisken, G.: A distance comparison principle for evolving curves. Asian J. Math. **2**(1), 127–133 (1998) MR1656553
16. Kim, I., Kwon, D.: Volume preserving mean curvature flow for star-shaped sets. [math.AP], (2018). arXiv:1808.04922v1 MR4083196
17. Lieberman, G.M.: Second order parabolic differential eqautions. World Scientific, Singapore (1996) MR1465184
18. Mantegazza, C.: Star-Shapedness under Mean Curvature Flow. Unpublished Notes, (2010) http://cvgmt.sns.it/HomePages/cm/other/MCF-1star.pdf
19. Mayer, U.F.: A singular example for the averaged mean curvature flow. Experiment. Math. **10**(1), 103–107 (2001) MR1822855
20. Schneider, R.: Convex bodies: the Brunn-Minkowski theory Encyclopedia of Mathematics and its Applications, 2nd edn. Cambridge University Press, Cambridge (2014) MR3155183
21. Privite communication with Professor Michael Gage
22. Smith, C.R.: A Characterization of Star-Shaped Sets. Am. Math. Mon. **75**(4), 386 (1968) MR0227724
23. Tsai, D.-H.: Geometric expansion of starshaped plane curves. Comm. Anal. Geom. **4**(3), 459–480 (1996) MR1415752
24. Tso, K.-S.: Deforming a hypersurface by its Gauss-Kronecker curvature. Comm. Pure Appl. Math. **38**(6), 867–882 (1985) MR0812353
25. Wang, X.-L., Wo, W.-F., Yang, M.: Evolution of non-simple closed curves in the area-preserving curvature flow. Proc. Roy. Soc. Edinburgh Sect. A **148**(3), 659–668 (2018) MR3826500
26. Yagisita, H.: Asymptotic behaviors of star-shaped curves expanding by $V = 1 - K$. Differ. Integr. Equ. **18**(2), 225–232 (2005) MR2106103

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.