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Star-shaped centrosymmetric curves under Gage's area-preserving flow.

(English. English summary)

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The paper under review concerns the theory of geometric evolution flows for closed planar curves. Particularly, the Gage area preserving flow is considered so that the evolution of a planar closed curve $X: S^1 \times [0, T) \rightarrow \mathbb{E}^2$ is governed by the equation

$$\begin{cases} \frac{\partial X}{\partial t}(\varphi, t) = \left(k(\varphi, t) - \frac{2\pi}{L(t)} \right) N(\varphi, t), \\ X(\varphi, 0) = X_0(\varphi), \end{cases}$$

where k , L and N stand for the curvature, the length and the unit inward normal of the evolving curve, respectively [see M. E. Gage, in *Nonlinear problems in geometry (Mobile, Ala., 1985)*, 51–62, Contemp. Math., 51, Amer. Math. Soc., Providence, RI, 1986; MR0848933].

If the initial curve $X_0: S^1 \rightarrow \mathbb{E}^2$ is assumed to be *convex*, then the flow exists for all $t \in [0, +\infty)$ so that the evolving curve converges to a circle. On the other hand, there are non-convex examples where the discussed convergence does not hold. Thus, the problem is to find natural classes of closed planar curves where the convergence of the Gage area preserving flow holds true.

The authors discuss the case of embedded *star-shaped* closed curves in \mathbb{E}^2 and prove the following statement as the main result.

Theorem. Let X_0 be a smooth embedded star-shaped closed curve in \mathbb{E}^2 . If X_0 is symmetric with respect to its star center O , then the Gage area preserving flow exists for all $t \in [0, +\infty)$ so that the evolving curve remains smooth, becomes convex in finite time and converges to a circle centered at O as $t \rightarrow +\infty$.

The question of whether the proved statement holds true without the extra symmetry requirement seems to be open. Actually, if X_0 is not assumed to be centrosymmetric, then the evolving curve might lose its star-shapedness, which would cause extra difficulties in studying its asymptotic behavior.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.