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Soliton solutions to the curve shortening flow on the 2-dimensional hyperbolic space. (English. English summary)

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According to the work of physicist P. A. Beck in the 1950s [in *Metal interfaces*, p. 208, Am. Soc. Test. Mater., 1952; per reviewer], the grain boundaries of a recrystallized metal, when annealed, migrate toward their centers of curvature. W. W. Mullins [J. Appl. Phys. **27** (1956), 900–904; MR0078836] considered the annealing model from mathematical viewpoints and presented three types of special solution to the evolution equation in the planar case. Mullins' solutions are called shrinkers, translators and rotators nowadays. The general metal annealing model is currently studied in mathematics as the mean curvature flow and, in the case of 1 dimension, it is called the curve shortening flow (CSF), formulated as equation (1.1) in the paper under review.

Let $X\colon I\times(a,b)\to M$ be a family of curves on a 2-dimensional Riemannian manifold. $X(\cdot,t)$ is called a solution to the CSF if it satisfies

$$\frac{\partial X}{\partial t}(\cdot,t)=\kappa(\cdot,t)N(\cdot,t)\quad \text{in } I\times(a,b),$$

where $\kappa(\cdot,t)$ is the geodesic curvature and $N(\cdot,t)$ the unit normal field. For mathematicians, the original motivation for studying the CSF was to find closed geodesics in Riemannian manifolds. Related results were given by M. A. Grayson [Ann. of Math. (2) **129** (1989), no. 1, 71–111; MR0979601], M. E. Gage [Indiana Univ. Math. J. **39** (1990), no. 4, 1037–1059; MR1087184], S. B. Angenent [Ann. of Math. (2) **162** (2005), no. 3, 1187–1241; MR2179729] and L. Ma and D. Chen [Ann. Mat. Pura Appl. (4) **186** (2007), no. 4, 663–684; MR2317784]. To fulfill these tasks, the asymptotic behaviors of the CSF had to be studied.

In fact, great progress was first made in the early 1980s by Gage [Duke Math. J. **50** (1983), no. 4, 1225–1229; MR0726325; Invent. Math. **76** (1984), no. 2, 357–364; MR0742856]. Later, the Gage-Hamilton-Grayson Theorem was proved [M. E. Gage and R. S. Hamilton, J. Differential Geom. **23** (1986), no. 1, 69–96; MR0840401; M. A. Grayson, J. Differential Geom. **26** (1987), no. 2, 285–314; MR0906392]. It says that a closed and embedded curve evolving by the CSF becomes a convex curve and then shrinks into a point which has asymptotically circular shape. However, given a generic, closed and immersed curve evolving according to the CSF, different kinds of singularities may occur. Angenent has classified the singularity model into two kinds [J. Differential Geom. **33** (1991), no. 3, 601–633; MR1100205]. He showed that the type I singularity models are self-similar shrinkers, which have been studied by U. Abresch and J. C. Langer [J. Differential Geom. **23** (1986), no. 2, 175–196; MR0845704] and C. L. Epstein and M. I. Weinstein [Comm. Pure Appl. Math. **40** (1987), no. 1, 119–139; MR0865360]. Moreover, the type II singularity models were proved to be translators, whose name "Grim Reaper" is said to have been given by the legendary mathematician E. Calabi. Angenent's result has been generalized by S. J. Altschuler [J. Differential Geom. **34** (1991), no. 2, 491–514; MR1131441] and Y. Y. Yang and X. Jiao [Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 4, 715–722; MR2156947] in higher-dimensional Euclidean space. Singularity analysis of the CSF on a Riemannian surface was given by D. L. Johnson and M. Muraleetharan [Int. J. Pure Appl. Math. **61** (2010), no. 2, 121–146; MR2668967], and on a general Riemannian manifold by S. J. Pan [Acta Math. Sin. (Engl. Ser.) **37** (2021), no. 11, 1783–1793; MR4344567].

To further study the CSF, it is natural to investigate general soliton solutions. H. P. Halldorsson classified all self-similar solutions on the Euclidean plane [Trans. Amer.

Math. Soc. **364** (2012), no. 10, 5285–5309; MR2931330]. H. F. Santos dos Reis and K. Tenenblat dealt with the case of the sphere [Proc. Amer. Math. Soc. **147** (2019), no. 11, 4955–4967; MR4011527]. The authors of the paper under review consider the classification problem on the hyperbolic plane. Together with the result by E. Woolgar and R. Xie [Proc. Amer. Math. Soc. **150** (2022), no. 3, 1301–1319; MR4375723], these excellent studies help us to systematically understand the asymptotic behavior of the CSF in 2-dimensional space forms. *Laiyuan Gao*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.