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A fully discrete difference scheme for a diffusion-wave system

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Abstract

A fully discrete difference scheme is derived for a diffusion-wave system by introducing two new variables to transform the original equation into a low order system of equations. The solvability, stability and L_∞ convergence are proved by the energy method. Similar results are provided for a slow diffusion system. A numerical example demonstrates the theoretical results.

Keywords: Diffusion-wave system; Finite difference; Convergence; Solvability; Stability

1. Introduction

This article is concerned with a numerical solution to a fractional diffusion-wave (FDW) system subjected to a non-homogeneous field. A fractional diffusion-wave equation is a linear integro-partial differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order $\alpha > 0$ [13]. Oldham and Spanier [20] considered a fractional diffusion equation that contains first order derivative in space and half order derivative in time. Nigmatullin [18,19] pointed out that many of the universal electromagnetic, acoustic, and

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mechanical responses can be modelled accurately using the FDW equations. Wess [24] and Schneider and Wess [22] presented solutions of FDW equations in terms of Fox's H-functions. Fujita [5] presented the existence and the uniqueness of the solution of the Cauchy problem of the following type: $\partial^\alpha u(x,t)/\partial t^\alpha = \partial^\beta u(x,t)/\partial x^\beta$, $1 \leqslant \alpha, \beta \leqslant 2$. The results presented offer an interpretation to phenomena between the heat equation ($\alpha = 1,\ \beta = 2$) and the wave equation ($\alpha = 2,\ \beta = 2$). Fujita [6,7] considered integro-differential equations which exhibit heat diffusion and wave propagation properties. He also demonstrated that certain operators associated with the equations can be decomposed and the solutions can be written as the sum of the solutions of the decomposed operators.

Ginoa et al. [8] presented a fractional diffusion equation describing relaxation phenomena in viscoelastic materials. Mbodje and Montseny [16] investigated the existence, uniqueness, and asymptotic decay of the wave equation with fractional derivative feedback, and showed that the method developed can be easily adapted to a wide class of problems involving fractional derivative or integral operators of the time variable.

Mainardi [11,12] used Laplace transform method to obtain the fundamental solution of the FDW equations and expressed them in terms of auxiliary function $M(z,\beta)$, where $z = |x|/t^\beta$ is the similarity variable. He further showed that such a function is an entire function of Wright type. Agrawal [1,2] presented a general solution to FDW equations containing fourth order space derivative defined in unbounded and bounded domains.

Metzler and Klafter [17] used Fourier–Laplace transform and the separation of variables to solve the fractional diffusion equation for absorbing and reflecting boundary value problems. Helfer [9] presented the solution of a fractional diffusion problem in terms of H-functions. Mainardi et al. [14] presented the fundamental solution (Green function) for the space–time fractional diffusion equation, which is obtained from the standard diffusion equation by replacing the second-order space derivative with a Riesz–Feller derivative of order $\alpha \in (0,2]$ and skewness θ ($|\theta| \leqslant \min\{\alpha, 2-\alpha\}$), and the first-order time derivative with a fractional derivative of order $\beta \in (0,2]$. They also presented explicit formulae for various functions in terms of parameters α, θ, and β.

Agrawal [3] used the method of separation of variables to identify the eigenfunctions and to reduce the differential equation of an FDW into a set of infinite equations each of which describes the dynamics of an eigenfunction. A Laplace transform technique is used to obtain the fractional Green's function and a Duhamel integral type expression for the system's response.

Compared with the considerable work on the theoretical analysis, only a little work has been done on the numerical method. Sanz-Serna [21] presented a temporal semi-discrete algorithm and proved the one order convergence. The linear equation investigated by him could be considered as a $3/2$ order time-fractional diffusion-wave equation. Same problems are investigated by Lopez-Marcos [10] and Tang [23]. A backward-Euler scheme and a Crank–Nicolson scheme are presented in [10] and [23], respectively. The stability and convergence are obtained. Bechelova [4] proposed a difference scheme for the mixed boundary value problem of an α ($0 < \alpha < 1$) order time fractional diffusion equation (a slow diffusion system) and proved the $\mathrm{O}(\tau^\alpha + h^2)$ order conditional convergence in the uniform metric by the maximum principle.

In this article, we give a fully discrete difference scheme for the FDW equation and prove that the difference scheme is uniquely solvable, unconditionally stable and convergent in L_∞ norm. The convergence order is $\mathrm{O}(\tau^{3-\alpha} + h^2)$.

Consider the FDW equation [3]

$$\frac{1}{c}\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{1}{K}f(x,t),\quad 0\leqslant x\leqslant L,\ t>0 \tag{1.1}$$

along with the initial conditions

$$u(x,0)=\phi(x),\quad \frac{\partial u(x,0)}{\partial t}=\psi(x),\quad 0\leqslant x\leqslant L \tag{1.2}$$

and the boundary conditions

$$u(0,t)=0,\quad u(L,t)=0,\quad t>0, \tag{1.3}$$

where c and K are constants of dimensions $Length^2Time^{-\alpha}$ and $Length^2$, $x\in[0,L]$ and $t>0$ are space and time variables, $\phi(0)=\phi(L)=0$, $u=u(x,t)$ and $f(x,t)$ are the field variables, and

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{\partial^2 u(x,s)}{\partial s^2}\frac{\mathrm{d}s}{(t-s)^{\alpha-1}},\quad 1<\alpha<2$$

with Γ denoting the gamma function. When $\alpha=1$, Eq. (1.1) represents a diffusion equation, c and $f(x,t)$ are called the diffusion coefficient and the source term, respectively. When $\alpha=2$, Eq. (1.1) represents a wave equation, c and $f(x,t)$ denote the square of the wave velocity and an external force field, respectively. For $1<\alpha<2$ the fractional equation in (1.1) is expected to interpolate the diffusion equation and the wave equation, thus in this case it could be referred to as the time-fractional diffusion-wave equation. We have to point out that for $1<\alpha<2$ both the initial conditions in (1.2) are necessary as the wave equation ($\alpha=2$) but for $0<\alpha<1$ only initial condition $u(x,0)=\phi(x)$ can be imposed as the diffusion equation ($\alpha=1$) (see, e.g., [15,3]).

Let $\omega_h\equiv\{x_i\mid 0\leqslant i\leqslant M\}$ is a uniform mesh of the interval $[0,L]$, where $x_i=ih,\ 0\leqslant i\leqslant M$ with $h=L/M$. Let $\omega_\tau\equiv\{t_n\mid n\geqslant 0\}$, where $t_n=n\tau,\ \tau>0$.

Suppose $u=\{u_i^n\mid 0\leqslant i\leqslant M,\ n\geqslant 0\}$ is a grid function on $\omega_h\times\omega_\tau$. Introduce the following notations:

$$u_i^{n-1/2}=\frac{1}{2}\left(u_i^n+u_i^{n-1}\right),\qquad \delta_t u_i^{n-1/2}=\frac{1}{\tau}\left(u_i^n-u_i^{n-1}\right),$$

$$\delta_x u_{i-1/2}^n=\frac{1}{h}\left(u_i^n-u_{i-1}^n\right),\qquad \delta_x^2 u_i^n=\frac{1}{h}\left(\delta_x u_{i+1/2}^n-\delta_x u_{i-1/2}^n\right),$$

where $u_i^{n-1/2}$ is an average of u at the points (x_i,t_n) and (x_i,t_{n-1}) and $\delta_t u_i^{n-1/2}$ is the difference quotient of u based on these two points; $\delta_x u_{i-1/2}^n$ is the first-order difference quotient of u on the points (x_i,t_n) and (x_{i-1},t_n), and $\delta_x^2 u_i^n$ is the second-order difference quotient at the points (x_{i-1},t_n), (x_i,t_n) and (x_{i+1},t_n). We also define

$$\|u^n\|_\infty=\max_{0\leqslant i\leqslant M}|u_i^n|,\qquad |\delta_x u^n|=\sqrt{h\sum_{i=1}^M\left(\delta_x u_{i-1/2}^n\right)^2}.$$

In addition, if $u_0^n=0$ and $u_M^n=0$, we have

$$\|u^n\|_\infty\leqslant\frac{\sqrt{L}}{2}|\delta_x u^n|. \tag{1.4}$$

The difference scheme we will consider for (1.1)–(1.3) is as follows:

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0\delta_t u_i^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t u_i^{k-1/2}-a_{n-1}\psi_i\right]=\delta_x^2 u_i^{n-1/2}+\frac{1}{K}f_i^{n-1/2},$$
$$1\leqslant i\leqslant M-1,\ n\geqslant 1, \tag{1.5}$$
$$u_i^0=\phi_i,\quad 0\leqslant i\leqslant M, \tag{1.6}$$
$$u_0^n=0,\quad u_M^n=0,\quad n\geqslant 1, \tag{1.7}$$

where

$$a_l\equiv\int_{t_l}^{t_{l+1}}\frac{\mathrm{d}t}{t^{\alpha-1}}=\frac{1}{2-\alpha}\left[(t_{l+1})^{2-\alpha}-(t_l)^{2-\alpha}\right]=\frac{\tau^{2-\alpha}}{2-\alpha}\left[(l+1)^{2-\alpha}-l^{2-\alpha}\right],\quad l\geqslant 0, \tag{1.8}$$

and

$$\phi_i=\phi(x_i),\quad 0\leqslant i\leqslant M;\quad \psi_i=\psi(x_i),\quad f_i^{n-1/2}=f\left(x_i,\frac{t_n+t_{n-1}}{2}\right),\quad 1\leqslant i\leqslant M-1,\ n\geqslant 1.$$

It is easy to know that

$$a_0=\frac{\tau^{2-\alpha}}{2-\alpha},\qquad a_l>a_{l+1},\quad l\geqslant 0, \tag{1.9}$$

and

$$\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})=a_0-a_{n-1},\quad n\geqslant 2. \tag{1.10}$$

At each time level, (1.5)–(1.7) is a tridiagonal system of linear algebraic equations, which can be solved by the double sweep method (Thomas algorithm). In the following, for simplicity, we suppose that problem (1.1)–(1.3) has solution $u(x,t)\in C_{x,t}^{4,3}([0,L]\times[0,\infty))$.

The remainder of the article is arranged as follows. In Section 2, the difference scheme (1.5)–(1.7) is derived by introducing two new variables and transforming the original equation (1.1) into a low order system of equations. In Section 3, the main results are proved, which are Theorems 3.2 and 3.3. In Section 4, similar results are presented for a slow diffusion system. Section 5 provides a numerical example to demonstrate the theoretical results.

2. The derivation of the difference scheme

For the derivation of the difference scheme, we need the following lemmas.

Lemma 2.1. *For $n\geqslant 1$ and $t_k=k\tau$, $0\leqslant k\leqslant n$, we have*

$$0\leqslant\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{(t_n-t)^{2-\alpha}-\left[\frac{t-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha}+\frac{t_k-t}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\}\mathrm{d}t$$
$$\leqslant\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-\left(1+2^{1-\alpha}\right)\right]\tau^{3-\alpha}.$$

Proof. Let $g(t)=(t_n-t)^{2-\alpha}$, then

$$g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]=\frac{1}{2}g''(\xi_k)(t-t_k)(t-t_{k-1})$$
$$=\frac{1}{2}(2-\alpha)(\alpha-1)(t_n-\xi_k)^{-\alpha}(t_k-t)(t-t_{k-1})\geqslant 0,$$

where $\xi_k\in(t_{k-1},t_k)$, $t\in(t_{k-1},t_k)$. From the above inequality, we have

$$\sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k}\left\{g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}\mathrm{d}t$$
$$=\sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k}\frac{1}{2}(2-\alpha)(\alpha-1)(t_n-\xi_k)^{-\alpha}(t_k-t)(t-t_{k-1})\,\mathrm{d}t$$
$$\leqslant\frac{1}{2}(2-\alpha)(\alpha-1)\sum_{k=1}^{n-2}(t_n-t_k)^{-\alpha}\int_{t_{k-1}}^{t_k}(t_k-t)(t-t_{k-1})\,\mathrm{d}t.$$

Since $\int_{t_{k-1}}^{t_k}(t_k-t)(t-t_{k-1})\,\mathrm{d}t=\tau^3/6$, we obtain

$$\sum_{k=1}^{n-2}\int_{t_{k-1}}^{t_k}\left\{g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}\mathrm{d}t$$
$$\leqslant\frac{1}{12}(2-\alpha)(\alpha-1)\tau^3\sum_{k=1}^{n-2}(t_n-t_k)^{-\alpha}\leqslant\frac{1}{12}(2-\alpha)(\alpha-1)\tau^2\int_{t_1}^{t_{n-1}}(t_n-t)^{-\alpha}\,\mathrm{d}t$$
$$=\frac{1}{12}(2-\alpha)\tau^2\left[(t_n-t_{n-1})^{1-\alpha}-(t_n-t_1)^{1-\alpha}\right]\leqslant\frac{2-\alpha}{12}\tau^{3-\alpha}. \tag{2.1}$$

On the other hand,

$$\sum_{k=n-1}^{n}\int_{t_{k-1}}^{t_k}\left\{g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}\mathrm{d}t$$
$$=\int_{t_{n-2}}^{t_n}g(t)\,\mathrm{d}t-\left[\frac{1}{2}g(t_{n-2})+g(t_{n-1})\right]\tau$$
$$=\int_{t_{n-2}}^{t_n}(t_n-t)^{2-\alpha}\,\mathrm{d}t-\left[\frac{1}{2}(t_n-t_{n-2})^{2-\alpha}+(t_n-t_{n-1})^{2-\alpha}\right]\tau$$
$$=\left[\frac{2^{3-\alpha}}{3-\alpha}-\left(1+2^{1-\alpha}\right)\right]\tau^{3-\alpha}. \tag{2.2}$$

The lemma follows from (2.1) and (2.2):

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}\mathrm{d}t$$
$$=\left(\sum_{k=1}^{n-2}+\sum_{k=n-1}^{n}\right)\int_{t_{k-1}}^{t_k}\left\{g(t)-\left[\frac{t-t_{k-1}}{\tau}g(t_k)+\frac{t_k-t}{\tau}g(t_{k-1})\right]\right\}\mathrm{d}t$$
$$\leqslant\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-\left(1+2^{1-\alpha}\right)\right]\tau^{3-\alpha}. \qquad \square$$

Lemma 2.2. *Suppose* $g(t)\in C^2[0,t_n]$. *Then*

$$\left|\int_0^{t_n}g'(t)\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}-\sum_{k=1}^{n}\frac{g(t_k)-g(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}\right|$$
$$\leqslant\frac{1}{2-\alpha}\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-\left(1+2^{1-\alpha}\right)\right]\max_{0\leqslant t\leqslant t_n}\left|g''(t)\right|\tau^{3-\alpha}.$$

Proof. For simplicity, denote

$$A\equiv\int_0^{t_n}g'(t)\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}-\sum_{k=1}^{n}\frac{g(t_k)-g(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}.$$

Using Taylor expansion with integral remainder, we have

$$g'(t)-\frac{g(t_k)-g(t_{k-1})}{\tau}=\frac{1}{\tau}\left[\int_{t_{k-1}}^{t}g''(s)(s-t_{k-1})\,\mathrm{d}s-\int_{t}^{t_k}g''(s)(t_k-s)\,\mathrm{d}s\right],$$

which yields

$$A=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left[g'(t)-\frac{g(t_k)-g(t_{k-1})}{\tau}\right]\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}$$
$$=\frac{1}{\tau}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left[\int_{t_{k-1}}^{t}g''(s)(s-t_{k-1})\,\mathrm{d}s-\int_{t}^{t_k}g''(s)(t_k-s)\,\mathrm{d}s\right]\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}.$$

Exchanging the order of integration, we get

$$A=\frac{1}{2-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{(t_n-s)^{2-\alpha}-\left[\frac{s-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha}+\frac{t_k-s}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\}g''(s)\,\mathrm{d}s.$$

Applying Lemma 2.1, we obtain the result:

$$
\begin{aligned}
|A| &\leqslant \frac{1}{2-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{(t_n-s)^{2-\alpha}-\left[\frac{s-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha}+\frac{t_k-s}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\}\left|g''(s)\right|\,\mathrm{d}s\\
&\leqslant \frac{1}{2-\alpha}\max_{t_0\leqslant t\leqslant t_n}\left|g''(t)\right|\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left\{(t_n-s)^{2-\alpha}-\left[\frac{s-t_{k-1}}{\tau}(t_n-t_k)^{2-\alpha}+\frac{t_k-s}{\tau}(t_n-t_{k-1})^{2-\alpha}\right]\right\}\mathrm{d}s\\
&\leqslant \frac{1}{2-\alpha}\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-\left(1+2^{1-\alpha}\right)\right]\max_{0\leqslant t\leqslant t_n}\left|g''(t)\right|\tau^{3-\alpha}. \qquad \square
\end{aligned}
$$

Lemma 2.3. *Suppose* $g(t)\in C^2[0,t_n]$. *Then*

$$
\begin{aligned}
&\left|\int_0^{t_n} g'(t)\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}-\frac{1}{\tau}\left[a_0g(t_n)-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})g(t_k)-a_{n-1}g(t_0)\right]\right|\\
&\quad\leqslant \frac{1}{2-\alpha}\left[\frac{2-\alpha}{12}+\frac{2^{3-\alpha}}{3-\alpha}-\left(1+2^{1-\alpha}\right)\right]\max_{0\leqslant t\leqslant t_n}\left|g''(t)\right|\tau^{3-\alpha},
\end{aligned}
$$

where a_l *is defined in* (1.8) *and* $1<\alpha<2$.

Proof. Observing Lemma 2.2, it suffices to verify

$$
\sum_{k=1}^{n}\frac{g(t_k)-g(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}=\frac{1}{\tau}\left[a_0g(t_n)-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})g(t_k)-a_{n-1}g(t_0)\right].
$$

In fact,

$$
\begin{aligned}
&\sum_{k=1}^{n}\frac{g(t_k)-g(t_{k-1})}{\tau}\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}\\
&\quad=\sum_{k=1}^{n}\frac{g(t_k)-g(t_{k-1})}{\tau}\frac{1}{2-\alpha}\left[(t_{n-k+1})^{2-\alpha}-(t_{n-k})^{2-\alpha}\right]=\frac{1}{\tau}\sum_{k=1}^{n}a_{n-k}\left(g(t_k)-g(t_{k-1})\right)\\
&\quad=\frac{1}{\tau}\left[a_0g(t_n)-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})g(t_k)-a_{n-1}g(t_0)\right].
\end{aligned}
$$

This completes the proof. $\square$

Let

$$
v(x,t)=\frac{\partial u(x,t)}{\partial t} \tag{2.3}
$$

and

$$w(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{\partial v(x,s)}{\partial s}\frac{\mathrm{d}s}{(t-s)^{\alpha-1}}. \tag{2.4}$$

Then (1.1) becomes

$$\frac{1}{c}w(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{1}{K}f(x,t). \tag{2.5}$$

Define the grid functions

$$U_i^n=u(x_i,t_n),\quad V_i^n=v(x_i,t_n),\quad W_i^n=w(x_i,t_n),\quad 0\leqslant i\leqslant M,\ n\geqslant 0.$$

Using Taylor expansion, it follows from (2.3) and (2.5) that

$$V_i^{n-1/2}=\delta_t U_i^{n-1/2}+(r_1)_i^{n-1/2}, \tag{2.6}$$

and

$$\frac{1}{c}W_i^{n-1/2}=\delta_x^2 U_i^{n-1/2}+\frac{1}{K}f_i^{n-1/2}+(r_2)_i^{n-1/2}, \tag{2.7}$$

and there exists a constant c_1 such that

$$\left|(r_1)_i^{n-1/2}\right|\leqslant c_1\tau^2 \tag{2.8}$$

and

$$\left|(r_2)_i^{n-1/2}\right|\leqslant c_1\left(\tau^2+h^2\right). \tag{2.9}$$

Based on Lemma 2.3, we have

$$\begin{aligned}W_i^n&=\frac{1}{\Gamma(2-\alpha)}\int_0^{t_n}\frac{\partial v(x_i,t)}{\partial t}\frac{\mathrm{d}t}{(t_n-t)^{\alpha-1}}\\&=\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0V_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})V_i^k-a_{n-1}V_i^0\right]+\mathrm{O}\left(\tau^{3-\alpha}\right),\quad n\geqslant 0.\end{aligned}$$

Consequently,

$$\begin{aligned}W_i^{n-1/2}&=\frac{1}{2}\left(W_i^n+W_i^{n-1}\right)\\&=\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0V_i^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})V_i^{k-1/2}-a_{n-1}V_i^0\right]+(r_3)_i^{n-1/2}\end{aligned} \tag{2.10}$$

and there exists a constant c_2 such that

$$\left|(r_3)_i^{n-1/2}\right|\leqslant c_2\tau^{3-\alpha}. \tag{2.11}$$

Substituting (2.6) into (2.10), we have

$$W_i^{n-1/2} = \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0\delta_t U_i^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t U_i^{k-1/2} - a_{n-1}V_i^0\right]$$
$$+ \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0(r_1)_i^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(r_1)_i^{k-1/2}\right] + (r_3)_i^{n-1/2}. \tag{2.12}$$

Then substituting above result into (2.7) and noticing $V_i^0 = v(x_i, 0) = \psi(x_i)$, we obtain

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0\delta_t U_i^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t U_i^{k-1/2} - a_{n-1}\psi(x_i)\right]$$
$$= \delta_x^2 U_i^{n-1/2} + \frac{1}{K} f_i^{n-1/2} + R_i^{n-1/2}, \quad 1 \leqslant i \leqslant M-1,\ n \geqslant 1, \tag{2.13}$$

where

$$R_i^{n-1/2} = -\frac{1}{c}\left\{\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0(r_1)_i^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(r_1)_i^{k-1/2}\right] + (r_3)_i^{n-1/2}\right\} + (r_2)_i^{n-1/2}.$$

According (1.9), (1.10), (2.8), (2.9) and (2.11), we have

$$\left|R_i^{n-1/2}\right| \leqslant \frac{1}{c}\left[\frac{2c_1}{(2-\alpha)\Gamma(2-\alpha)} + c_2 + cc_1\right]\left(h^2 + \tau^{3-\alpha}\right). \tag{2.14}$$

In addition, from (1.2), (1.3), we have

$$U_i^0 = \phi(x_i), \quad 0 \leqslant i \leqslant M \tag{2.15}$$

and

$$U_0^n = 0, \quad U_M^n = 0, \quad n \geqslant 1. \tag{2.16}$$

Observing (2.13) and (2.15), (2.16), it is natural to construct the difference scheme (1.5)–(1.7) for the problem (1.1)–(1.3).

3. Analysis of the difference scheme

Before we prove the solvability, stability and convergence, we give the following lemmas.

Lemma 3.1. *For any $G = \{G_1, G_2, G_3, \ldots\}$ and q, we have*

$$\sum_{n=1}^{N}\left[a_0 G_n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})G_k - a_{n-1}q\right]G_n \geqslant \frac{t_N^{1-\alpha}}{2}\tau\sum_{n=1}^{N}G_n^2 - \frac{t_N^{2-\alpha}}{2(2-\alpha)}q^2,$$
$$N = 1, 2, 3, \ldots,$$

where a_l is defined in (1.8).

Proof.

$$\sum_{n=1}^{N}\left[a_0G_n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})G_k-a_{n-1}q\right]G_n$$
$$=\sum_{n=1}^{N}a_0G_n^2-\sum_{n=2}^{N}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})G_kG_n-\sum_{n=1}^{N}a_{n-1}qG_n$$
$$\geqslant\sum_{n=1}^{N}a_0G_n^2-\frac{1}{2}\sum_{n=2}^{N}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\big(G_k^2+G_n^2\big)-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}\big(q^2+G_n^2\big)$$
$$=\sum_{n=1}^{N}a_0G_n^2-\frac{1}{2}\sum_{n=2}^{N}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})G_n^2-\frac{1}{2}\sum_{n=2}^{N}\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})G_k^2$$
$$-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}q^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}G_n^2.$$

Exchanging the summation order of the third term in the last inequality, we obtain

$$\sum_{n=1}^{N}\left[a_0G_n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})G_k-a_{n-1}q\right]G_n$$
$$\geqslant\sum_{n=1}^{N}a_0G_n^2-\frac{1}{2}\sum_{n=2}^{N}(a_0-a_{n-1})G_n^2-\frac{1}{2}\sum_{k=1}^{N-1}(a_0-a_{N-k})G_k^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}q^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}G_n^2$$
$$=\frac{1}{2}\sum_{n=1}^{N}a_{N-n}G_n^2-\frac{1}{2}\sum_{n=1}^{N}a_{n-1}q^2\geqslant\frac{1}{2}a_{N-1}\sum_{n=1}^{N}G_n^2-\frac{t_N^{2-\alpha}}{2(2-\alpha)}q^2$$
$$\geqslant\frac{1}{2}t_N^{1-\alpha}\tau\sum_{n=1}^{N}G_n^2-\frac{t_N^{2-\alpha}}{2(2-\alpha)}q^2,$$

where we have used that $\{a_l\}$ is a strictly decreasing sequence and

$$\sum_{n=1}^{N}a_{n-1}=\sum_{n=0}^{N-1}a_n=\int_{t_0}^{t_N}\frac{\mathrm{d}\xi}{\xi^{\alpha-1}}=\frac{t_N^{2-\alpha}}{2-\alpha},\qquad a_l=\int_{t_l}^{t_{l+1}}x^{1-\alpha}\,\mathrm{d}x\geqslant t_{l+1}^{1-\alpha}\tau.\qquad\square$$

Lemma 3.2. *Suppose* $\{u_i^n\}$ *is the solution of*

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0\delta_tu_i^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_tu_i^{k-1/2}-a_{n-1}q_i\right]=\delta_x^2u_i^{n-1/2}+P_i^{n-1/2},$$
$$1\leqslant i\leqslant M-1,\ n=1,2,\ldots,\tag{3.1}$$
$$u_i^0=\phi_i,\quad 0\leqslant i\leqslant M,\tag{3.2}$$
$$u_0^n=0,\quad u_M^n=0,\quad k=1,2,\ldots.\tag{3.3}$$

We have

$$\left|\delta_x u^n\right|^2 \leqslant \left|\delta_x u^0\right|^2 + \frac{t_n^{2-\alpha}}{c(2-\alpha)\Gamma(2-\alpha)} h\sum_{i=1}^{M-1} q_i^2 + c\Gamma(2-\alpha)t_n^{\alpha-1}\tau\sum_{k=1}^{n}\left[h\sum_{i=1}^{M-1}\left(P_i^{k-1/2}\right)^2\right],\quad n \geqslant 1. \tag{3.4}$$

Proof. Multiplying (3.1) by $h\tau\delta_t u_i^{n-1/2}$ and summing up for i from 1 to $M-1$ and for n from 1 to m, we obtain

$$\frac{1}{c\Gamma(2-\alpha)}\cdot h\sum_{i=1}^{M-1}\left\{\sum_{n=1}^{m}\left[a_0\delta_t u_i^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t u_i^{k-1/2} - a_{n-1}q_i\right]\delta_t u_i^{n-1/2}\right\}$$
$$= \tau\sum_{n=1}^{m}\left[h\sum_{i=1}^{M-1}\left(\delta_t u_i^{n-1/2}\right)\left(\delta_x^2 u_i^{n-1/2}\right)\right] + h\sum_{i=1}^{M-1}\left[\tau\sum_{n=1}^{m}\left(\delta_t u_i^{n-1/2}\right)P_i^{n-1/2}\right]. \tag{3.5}$$

Using Lemma 3.1, we have

$$h\sum_{i=1}^{M-1}\left\{\sum_{n=1}^{m}\left[a_0\delta_t u_i^{n-1/2} - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t u_i^{k-1/2} - a_{n-1}q_i\right]\delta_t u_i^{n-1/2}\right\}$$
$$\geqslant h\sum_{i=1}^{M-1}\left\{\frac{1}{2}t_m^{1-\alpha}\tau\sum_{n=1}^{m}\left(\delta_t u_i^{n-1/2}\right)^2 - \frac{t_m^{2-\alpha}}{2(2-\alpha)}q_i^2\right\}$$
$$\geqslant \frac{1}{2}t_m^{1-\alpha}\tau\sum_{n=1}^{m}\left\|\delta_t u^{n-1/2}\right\|^2 - \frac{t_m^{2-\alpha}}{2(2-\alpha)}h\sum_{i=1}^{M-1}q_i^2. \tag{3.6}$$

Applying (3.2) and (3.3), we have $\delta_t u_0^{n-1/2} = 0$ and $\delta_t u_M^{n-1/2} = 0$. Consequently,

$$\tau\sum_{n=1}^{m}\left[h\sum_{i=1}^{M-1}\left(\delta_t u_i^{n-1/2}\right)\delta_x^2 u_i^{n-1/2}\right]$$
$$= -\tau\sum_{n=1}^{m}\left[h\sum_{i=1}^{M}\left(\delta_x u_{i-1/2}^{n-1/2}\right)\delta_t\left(\delta_x u_{i-1/2}^{n-1/2}\right)\right] = -\frac{1}{2}\sum_{n=1}^{m}\left[h\sum_{i=1}^{M}\left(\delta_x u_{i-1/2}^{n}\right)^2 - h\sum_{i=1}^{M}\left(\delta_x u_{i-1/2}^{n-1}\right)^2\right]$$
$$= -\frac{1}{2}\left(\left|\delta_x u^m\right|^2 - \left|\delta_x u^0\right|^2\right). \tag{3.7}$$

In addition,

$$h\sum_{i=1}^{M-1}\left[\tau\sum_{n=1}^{m}\left(\delta_t u_i^{n-1/2}\right)P_i^{n-1/2}\right]$$
$$\leqslant h\sum_{i=1}^{M-1}\tau\sum_{n=1}^{m}\left[\frac{1}{c\Gamma(2-\alpha)}\frac{1}{2}t_m^{1-\alpha}\left(\delta_t u_i^{n-1/2}\right)^2 + \frac{1}{2}c\Gamma(2-\alpha)t_m^{\alpha-1}\left(P_i^{n-1/2}\right)^2\right]$$
$$\leqslant \frac{1}{c\Gamma(2-\alpha)}\frac{1}{2}t_m^{1-\alpha}\tau\sum_{n=1}^{m}\left\|\delta_t u^{n-1/2}\right\|^2 + \frac{1}{2}c\Gamma(2-\alpha)t_m^{\alpha-1}\tau\sum_{n=1}^{m}\left[h\sum_{i=1}^{M-1}\left(P_i^{n-1/2}\right)^2\right]. \tag{3.8}$$

Substituting (3.6)–(3.8) into (3.5), we obtain

$$\frac{1}{c\Gamma(2-\alpha)}\left\{\frac{1}{2}t_m^{1-\alpha}\tau\sum_{n=1}^{m}\left\|\delta_t u^{n-1/2}\right\|^2-\frac{t_m^{2-\alpha}}{2(2-\alpha)}h\sum_{i=1}^{M-1}q_i^2\right\}$$
$$\leqslant -\frac{1}{2}\left(\left|\delta_x u^m\right|^2-\left|\delta_x u^0\right|^2\right)+\frac{t_m^{1-\alpha}}{2c\Gamma(2-\alpha)}\tau\sum_{n=1}^{m}\left\|\delta_t u^{n-1/2}\right\|^2$$
$$+\frac{c}{2}\Gamma(2-\alpha)t_m^{\alpha-1}\tau\sum_{n=1}^{m}\left[h\sum_{i=1}^{M-1}\left(P_i^{n-1/2}\right)^2\right],$$

or

$$\left|\delta_x u^n\right|^2\leqslant\left|\delta_x u^0\right|^2+\frac{t_n^{2-\alpha}}{c(2-\alpha)\Gamma(2-\alpha)}h\sum_{i=1}^{M-1}q_i^2+c\Gamma(2-\alpha)t_n^{\alpha-1}\tau\sum_{k=1}^{n}\left[h\sum_{i=1}^{M-1}\left(P_i^{k-1/2}\right)^2\right],\quad n\geqslant 1.$$

This completes the proof. □

Theorem 3.1. *The difference scheme* (1.5)–(1.7) *is uniquely solvable.*

Proof. Since (1.5)–(1.7) is a system of linear algebraic equations at each time level, it suffices to show that the corresponding homogeneous equations:

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0\delta_t u_i^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t u_i^{k-1/2}\right]=\delta_x^2 u_i^{n-1/2},\quad 1\leqslant i\leqslant M-1,\ n\geqslant 1, \tag{3.9}$$

$$u_i^0=0,\quad 0\leqslant i\leqslant M, \tag{3.10}$$

$$u_0^n=0,\quad u_M^n=0,\quad n\geqslant 1, \tag{3.11}$$

have only zero solution. Using Lemma 3.2, we have

$$\left|\delta_x u^n\right|=0,\quad n=1,2,3,\ldots.$$

Combining the above equality with (3.11) arrives at

$$u_i^n=0,\quad 0\leqslant i\leqslant M,\ n\geqslant 1. \qquad \square$$

Theorem 3.2. *Let* $\{u_i^n \mid 0\leqslant i\leqslant M,\ n\geqslant 0\}$ *be the solution of the difference scheme* (1.5)–(1.7). *Then, we have*

$$\left|\delta_x u^n\right|^2\leqslant\left|\delta_x u^0\right|^2+\frac{t_n^{2-\alpha}}{c(2-\alpha)\Gamma(2-\alpha)}h\sum_{i=1}^{M-1}\psi_i^2+\frac{c\Gamma(2-\alpha)}{K^2}t_n^{\alpha-1}\tau\sum_{k=1}^{n}\left[h\sum_{i=1}^{M-1}\left(f_i^{k-1/2}\right)^2\right],\quad n\geqslant 1.$$

Proof. The result needed can be easily obtained from Lemma 3.2. □

Theorem 3.3. *Let* (1.1)–(1.3) *have solution* $u(x,t) \in C_{x,t}^{4,3}([0,L]\times[0,T])$ *and* $\{u_i^n \mid 0 \leqslant i \leqslant M,\ n \geqslant 0\}$ *be the solution of the difference scheme* (1.5)–(1.7)*. Then, for* $n\tau \leqslant T$*, we have*

$$\max_{0\leqslant i\leqslant M}\left|u(x_i,t_n)-u_i^n\right| \leqslant \frac{L}{2}\left[\frac{2c_1}{(2-\alpha)\Gamma(2-\alpha)}+c_2+cc_1\right]\sqrt{\frac{T^\alpha\Gamma(2-\alpha)}{c}}\left(h^2+\tau^{3-\alpha}\right).$$

Proof. Denote

$$\tilde{u}_i^n = U_i^n - u_i^n, \quad 0 \leqslant i \leqslant M,\ n \geqslant 0.$$

Subtracting (1.5)–(1.7) from (2.13), (2.15), (2.16), respectively, we obtain the error equations

$$\frac{1}{c\Gamma(2-\alpha)}\frac{1}{\tau}\left[a_0\delta_t\tilde{u}_i^{n-1/2}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\delta_t\tilde{u}_i^{k-1/2}\right]=\delta_x^2\tilde{u}_i^{n-1/2}+R_i^{n-1/2},$$
$$1\leqslant i\leqslant M-1,\ n\geqslant 1,$$
$$\tilde{u}_i^0=0, \quad 0\leqslant i\leqslant M,$$
$$\tilde{u}_0^n=0, \quad \tilde{u}_M^n=0, \quad n\geqslant 1.$$

Using Lemma 3.2, we have

$$\left|\delta_x\tilde{u}^n\right|^2 \leqslant c\Gamma(2-\alpha)t_n^{\alpha-1}\tau\sum_{k=1}^{n}\left[h\sum_{i=1}^{M-1}\left(R_i^{k-1/2}\right)^2\right], \quad n\geqslant 1.$$

Inserting (2.14) into the right hand of the above inequality, we get

$$\left|\delta_x\tilde{u}^n\right| \leqslant \left[\frac{2c_1}{(2-\alpha)\Gamma(2-\alpha)}+c_2+cc_1\right]\sqrt{\frac{Lt_n^\alpha\Gamma(2-\alpha)}{c}}\left(h^2+\tau^{3-\alpha}\right), \quad n\geqslant 1.$$

Noticing (1.4), we have the result:

$$\left\|\tilde{u}^n\right\|_\infty \leqslant \frac{L}{2}\left[\frac{2c_1}{(2-\alpha)\Gamma(2-\alpha)}+c_2+cc_1\right]\sqrt{\frac{T^\alpha\Gamma(2-\alpha)}{c}}\left(h^2+\tau^{3-\alpha}\right), \quad \text{when } n\tau\leqslant T. \qquad \square$$

4. A slow diffusion system

Consider the slow diffusion equation [3,4]

$$\frac{1}{c}\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{1}{K}f(x,t), \quad 0\leqslant x\leqslant L,\ t>0 \tag{4.1}$$

along with the initial condition

$$u(x,0)=\phi(x), \quad 0\leqslant x\leqslant L \tag{4.2}$$

and the boundary conditions

$$u(0,t)=0, \quad u(L,t)=0, \quad t>0, \tag{4.3}$$

where

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{\mathrm{d}s}{(t-s)^\alpha}, \quad 0 < \alpha < 1.$$

Let α be replaced by $\alpha + 1$ in Lemma 2.3 and in Lemma 3.1, we have

Lemma 4.1. *Suppose* $g(t) \in C^2[0, t_n]$. *Then*

$$\left| \int_0^{t_n} g'(t) \frac{\mathrm{d}t}{(t_n - t)^\alpha} - \frac{1}{\tau} \left[b_0 g(t_n) - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) g(t_k) - b_{n-1} g(t_0) \right] \right|$$
$$\leqslant \frac{1}{1-\alpha} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - \left(1 + 2^{-\alpha}\right) \right] \max_{0 \leqslant t \leqslant t_n} \left| g''(t) \right| \tau^{2-\alpha},$$

where $0 < \alpha < 1$ *and*

$$b_l \equiv \int_{t_l}^{t_{l+1}} \frac{\mathrm{d}t}{t^\alpha} = \frac{1}{1-\alpha} \left[(t_{l+1})^{1-\alpha} - (t_l)^{1-\alpha} \right] = \frac{\tau^{1-\alpha}}{1-\alpha} \left[(l+1)^{1-\alpha} - l^{1-\alpha} \right], \quad l \geqslant 0. \tag{4.4}$$

Lemma 4.2. *For any* $G = \{G_1, G_2, G_3, \ldots\}$ *and* q, *we have*

$$\sum_{n=1}^{N} \left[b_0 G_n - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) G_k - b_{n-1} q \right] G_n \geqslant \frac{t_N^{-\alpha}}{2} \tau \sum_{n=1}^{N} G_n^2 - \frac{t_N^{1-\alpha}}{2(1-\alpha)} q^2, \quad N = 1, 2, 3, \ldots,$$

where $0 < \alpha < 1$ *and* b_l *is defined in* (4.4).

Using Lemma 4.1 and similarly to the derivation of the difference scheme (1.5)–(1.7) for the problem (1.1)–(1.3) in Section 2, we may construct the following difference scheme for (4.1)–(4.3):

$$\frac{1}{c\Gamma(1-\alpha)} \frac{1}{\tau} \left[b_0 u_i^n - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) u_i^k - b_{n-1} u_i^0 \right] = \delta_x^2 u_i^n + \frac{1}{K} f_i^n,$$
$$1 \leqslant i \leqslant M-1, \; n \geqslant 1, \tag{4.5}$$

$$u_i^0 = \phi_i, \quad 0 \leqslant i \leqslant M, \tag{4.6}$$

$$u_0^n = 0, \quad u_M^n = 0, \quad n \geqslant 1. \tag{4.7}$$

Multiplying (4.5) by $h\tau u_i^n$, using Lemma 4.2 and similarly to the analysis in Section 3, we can prove the following theorem:

Theorem 4.1. *The difference scheme* (4.5)–(4.7) *is uniquely solvable and the following estimate is valid*:

$$\tau \sum_{k=1}^{n} \left| \delta_x u^k \right|^2 \leqslant \frac{t_n^{1-\alpha}}{2c(1-\alpha)\Gamma(1-\alpha)} h \sum_{i=1}^{M-1} \phi_i^2 + \frac{c\Gamma(1-\alpha) t_n^\alpha}{2K^2} \tau \sum_{k=1}^{n} \left[h \sum_{i=1}^{M-1} \left(f_i^k\right)^2 \right],$$
$$n \geqslant 1. \tag{4.8}$$

Furthermore, if (4.1)–(4.3) *has solution* $u(x,t) \in C^{4,2}_{x,t}([0,L]\times[0,T])$, *then the solution of the difference scheme* (4.5)–(4.7) *converges to the solution of the problem* (4.1)–(4.3) *with the convergence order of* $O(h^2+\tau^{2-\alpha})$ *in* L_∞*-norm in the sense that*

$$\sqrt{\tau\sum_{k=1}^{n}\big\|(U-u)^k\big\|_\infty^2} \leqslant C\big(h^2+\tau^{2-\alpha}\big), \quad \text{when } n\tau \leqslant T,$$

where C *is a constant independent of* h *and* τ.

5. Numerical example

In order to demonstrate the effectiveness of our difference scheme, we compute the following problem:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \sin(\pi x), \quad 0 \leqslant x \leqslant 1,\ 0 < t \leqslant 1, \tag{5.1}$$

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \leqslant x \leqslant 1, \tag{5.2}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 \leqslant t \leqslant 1. \tag{5.3}$$

The exact solution of the above problem is [3]

$$u(x,t) = \frac{1}{\pi^2}\big[1 - E_\alpha\big(-\pi^2 t^\alpha\big)\big]\sin(\pi x), \tag{5.4}$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

If $\alpha = 3/2$, then (5.4) can be expressed as follows:

$$u(x,t) = \frac{1}{\pi^2}\left[\frac{1}{\sqrt{\pi}}\sum_{m=1}^{\infty}\frac{(\pi^2 t^\alpha)^{2m-1}}{\prod_{i=1}^{3m-1}(i-\frac{1}{2})} - \sum_{m=1}^{\infty}\frac{(\pi^2 t^\alpha)^{2m}}{\prod_{i=1}^{3m} i}\right]\sin(\pi x). \tag{5.5}$$

Table 1
Some numerical results

$M \setminus (x,t)$	$(\frac{1}{8},1)$	$(\frac{2}{8},1)$	$(\frac{3}{8},1)$	$(\frac{4}{8},1)$
32	0.434346D−01	0.802566D−01	0.104860D+00	0.113500D+00
64	0.433077D−01	0.800222D−01	0.104554D+00	0.113168D+00
128	0.432653D−01	0.799438D−01	0.104452D+00	0.113058D+00
256	0.432510D−01	0.799174D−01	0.104417D+00	0.113020D+00
512	0.432461D−01	0.799084D−01	0.104405D+00	0.113008D+00
1024	0.432445D−01	0.799054D−01	0.104401D+00	0.113003D+00
Exact solution	0.432436D−01	0.799037D−01	0.104399D+00	0.113001D+00

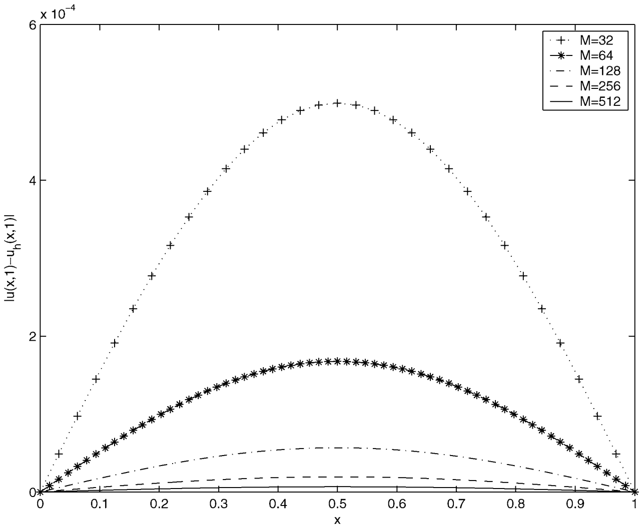

Fig. 1. The curves of the errors of the finite solutions at $t = 1$.

Table 2
The maximum errors $\|u - u_{h\tau}\|_\infty$

M	$\|u - u_{h\tau}\|_\infty$
32	0.4990405D−03
64	0.1675359D−03
128	0.5674313D−04
256	0.1940128D−04
512	0.6689013D−05
1024	0.2321606D−05

Take $\alpha = 3/2,\ h = \tau = 1/M$. Table 1 presents the numerical and exact solutions at some points for different mesh sizes. Fig. 1 plots the curves of the errors of the difference solutions on the line $t = 1$ for different mesh sizes. Table 2 gives the maximal errors of difference solutions at all mesh points for different mesh sizes. In Table 2, the maximal error is defined as follows

$$\|u - u_{h\tau}\|_\infty = \max_{1\leqslant n\leqslant M}\left\{\max_{0\leqslant i\leqslant M}\left|u(x_i, t_n) - u_i^n\right|\right\}.$$

It is clear that the finite difference solution is very accurate and converges quickly to the exact solution.

Suppose

$$\|u - u_{h\tau}\|_\infty \approx ch^p.$$

Then we have

$$-\log\|u - u_{h\tau}\|_\infty \approx -\log c + p(-\log h).$$

Using the data in Table 2 and with the help of MATLAB, we obtain linear fitting functions

$$-\log\|u - u_{h\tau}\|_\infty \approx 2.2476 + 1.5494(-\log h).$$

6. Conclusion

In this article, we present a difference scheme for the initial-boundary value problem of a diffusion-wave equation. The solvability, stability and convergence are proved by the energy method. Similar results are given for a slow diffusion system. A numerical example demonstrates the theoretical results.

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