3 Scalar Conservation Laws

We begin our study of conservation laws by considering the scalar case. Many of the difficulties encountered with systems of equations are already encountered here, and a good understanding of the scalar equation is required before proceeding.

3.1 The linear advection equation

We first consider the linear advection equation, derived in Chapter 2, which we now write as

$$u_t + au_x = 0. \tag{3.1}$$

The Cauchy problem is defined by this equation on the domain $-\infty < x < \infty$, $t \geq 0$ together with initial conditions

$$u(x,0) = u_0(x). \tag{3.2}$$

As noted previously, the solution is simply

$$u(x,t) = u_0(x - at) \tag{3.3}$$

for $t \geq 0$. As time evolves, the initial data simply propagates unchanged to the right (if $a > 0$) or left (if $a < 0$) with velocity a. The solution $u(x,t)$ is constant along each ray $x - at = x_0$, which are known as the **characteristics** of the equation. (See Fig. 3.1 for the case $a > 0$.)

Note that the characteristics are curves in the x-t plane satisfying the ordinary differential equations $x'(t) = a,\ x(0) = x_0$. If we differentiate $u(x,t)$ along one of these curves to find the rate of change of u along the characteristic, we find that

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \frac{\partial}{\partial t}u(x(t),t) + \frac{\partial}{\partial x}u(x(t),t)\,x'(t) \\ &= u_t + au_x \\ &= 0, \end{aligned} \tag{3.4}$$

confirming that u is constant along these characteristics.

More generally, we might consider a variable coefficient advection equation of the form

$$u_t + (a(x)u)_x = 0, \tag{3.5}$$

where $a(x)$ is a smooth function. Recalling the derivation of the advection equation in Chapter 2, this models the evolution of a chemical concentration $u(x,t)$ in a stream with variable velocity $a(x)$.

We can rewrite (3.5) as

$$u_t + a(x)u_x = -a'(x)u \tag{3.6}$$

or

$$\left(\frac{\partial}{\partial t} + a(x)\frac{\partial}{\partial x}\right) u(x,t) = -a'(x)u(x,t). \tag{3.7}$$

It follows that the evolution of u along any curve $x(t)$ satisfying

$$\begin{aligned} x'(t) &= a(x(t)) \\ x(0) &= x_0 \end{aligned} \tag{3.8}$$

satisfies a simple ordinary differential equation (ODE):

$$\frac{d}{dt}u(x(t),t) = -a'(x(t))\,u(x(t),t). \tag{3.9}$$

The curves determined by (3.8) are again called characteristics. In this case the solution $u(x,t)$ is not constant along these curves, but can be easily determined by solving two sets of ODEs.

It can be shown that if $u_0(x)$ is a smooth function, say $u_0 \in C^k(-\infty,\infty)$, then the solution $u(x,t)$ is equally smooth in space and time, $u \in C^k((-\infty,\infty) \times (0,\infty))$.

3.1.1 Domain of dependence

Note that solutions to the linear advection equations (3.1) and (3.5) have the following property: the solution $u(x,t)$ at any point $(\bar{x},\bar{t})$ depends on the initial data u_0 only at a *single* point, namely the point $\bar{x}_0$ such that $(\bar{x},\bar{t})$ lies on the characteristic through $\bar{x}_0$. We could change the initial data at any points other than $\bar{x}_0$ *without* affecting the solution $u(\bar{x},\bar{t})$. The set $\mathcal{D}(\bar{x},\bar{t}) = \{\bar{x}_0\}$ is called the **domain of dependence** of the point $(\bar{x},\bar{t})$. Here this domain consists of a single point. For a system of equations this domain is typically an interval, but a fundamental fact about hyperbolic equations is that it is always a *bounded* interval. The solution at $(\bar{x},\bar{t})$ is determined by the initial data within some finite distance of the point $\bar{x}$. The size of this set usually increases with $\bar{t}$, but we have a bound of the form $\mathcal{D} \subset \{x : |x-\bar{x}| \le a_{\max}\bar{t}\}$ for some value $a_{\max}$. Conversely, initial data at any given point x_0 can influence the solution only within some

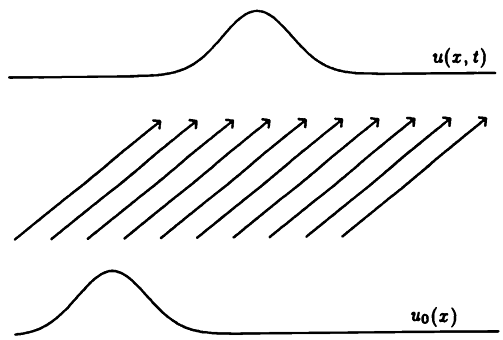

Figure 3.1. Characteristics and solution for the advection equation.

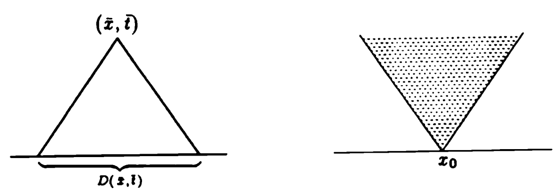

Figure 3.2. Domain of dependence and range of influence.

cone $\{x : |x - x_0| \leq a_{\max}t\}$ of the x-t plane. This region is called the **range of influence** of the point x_0. See Figure 3.2 for an illustration. We summarize this by saying that hyperbolic equations have **finite propagation speed**; information can travel with speed at most $a_{\max}$. This has important consequences in developing numerical methods.

3.1.2 Nonsmooth data

In the manipulations performed above, we have assumed differentiability of $u(x,t)$. However, from our observation that the solution along a characteristic curve depends only on the one value $u_0(x_0)$, it is clear that spatial smoothness is not required for this construction of $u(x,t)$ from $u_0(x)$. We can thus define a "solution" to the PDE even if $u_0(x)$ is not a smooth function. Note that if $u_0(x)$ has a singularity at some point x_0 (a discontinuity in u_0 or some derivative), then the resulting $u(x,t)$ will have a singularity of the same order along the characteristic curve through x_0, but will remain smooth along characteristics through smooth portions of the data. This is a fundamental property of *linear* hyperbolic equations: singularities propagate only along characteristics.

If u_0 is nondifferentiable at some point then $u(x,t)$ is no longer a classical solution of the differential equation everywhere. However, this function *does* satisfy the integral form of the conservation law, which continues to make sense for nonsmooth u. Recall that the integral form is more fundamental physically than the differential equation, which was derived from the integral form under the additional assumption of smoothness. It thus makes perfect sense to accept this generalized solution as solving the conservation law.

EXERCISE 3.1. *Let $f(u) = au$, with a constant, and let $u_0(x)$ be any integrable function. Verify that the function $u(x,t) = u_0(x-at)$ satisfies the integral form (2.16) for any x_1, x_2, t_1 and t_2.*

Other approaches can also be taken to defining this generalized solution, which extend better to the study of nonlinear equations where we can no longer simply integrate along characteristics.

One possibility is to approximate the nonsmooth data $u_0(x)$ by a sequence of smooth functions $u_0^\epsilon(x)$, with

$$\|u_0 - u_0^\epsilon\|_1 < \epsilon$$

as $\epsilon \to 0$. Here $\|\cdot\|_1$ is the 1-norm, defined by

$$\|v\|_1 = \int_{-\infty}^{\infty} |v(x)|\, dx. \tag{3.10}$$

For the linear equation we know that the PDE together with the smooth data u_0^ϵ has a smooth classical solution $u^\epsilon(x,t)$ for all $t \geq 0$. We can now define the generalized solution $u(x,t)$ by taking the limit of $u^\epsilon(x,t)$ as $\epsilon \to 0$. For example, the constant coefficient problem (3.1) has classical smooth solutions

$$u^\epsilon(x,t) = u_0^\epsilon(x - at)$$

and clearly at each time t the 1-norm limit exists and satisfies

$$u(x,t) = \lim_{\epsilon \to 0} u_0^\epsilon(x - at) = u_0(x - at)$$

as expected.

Unfortunately, this approach of smoothing the initial data will not work for nonlinear problems. As we will see, solutions to the nonlinear problem can develop singularities even if the initial data is smooth, and so there is no guarantee that classical solutions with data $u_0^\epsilon(x)$ will exist.

A better approach, which does generalize to nonlinear equations, is to leave the initial data alone but modify the PDE by adding a small diffusive term. Recall from Chapter 2 that the conservation law (3.1) should be considered as an approximation to the advection-diffusion equation

$$u_t + au_x = \epsilon u_{xx} \tag{3.11}$$

for ϵ very small. If we now let $u^\epsilon(x,t)$ denote the solution of (3.11) with data $u_0(x)$, then $u^\epsilon \in C^\infty((-\infty,\infty)\times(0,\infty))$ even if $u_0(x)$ is not smooth since (3.11) is a parabolic equation. We can again take the limit of $u^\epsilon(x,t)$ as $\epsilon \to 0$, and will obtain the same generalized solution $u(x,t)$ as before.

Note that the equation (3.11) simplifies if we make a change of variables to follow the characteristics, setting

$$v^\epsilon(x,t) = u^\epsilon(x + at, t).$$

Then it is easy to verify that v^ϵ satsifies the **heat equation**

$$v_t^\epsilon(x,t) = \epsilon v_{xx}^\epsilon(x,t). \tag{3.12}$$

Using the well-known solution to the heat equation to solve for $v(x,t)$, we have $u^\epsilon(x,t) = v^\epsilon(x - at, t)$ and so can explicitly calculate the "vanishing viscosity" solution in this case.

EXERCISE 3.2. *Show that the vanishing viscosity solution* $\lim_{\epsilon\to 0} u^\epsilon(x,t)$ *is equal to* $u_0(x - at)$.

3.2 Burgers' equation

Now consider the nonlinear scalar equation

$$u_t + f(u)_x = 0 \tag{3.13}$$

where $f(u)$ is a nonlinear function of u. We will assume for the most part that $f(u)$ is a convex function, $f''(u) > 0$ for all u (or, equally well, f is concave with $f''(u) < 0\ \forall u$). The convexity assumption corresponds to a "genuine nonlinearity" assumption for systems of equations that holds in many important cases, such as the Euler equations. The nonconvex case is also important in some applications (e.g. oil reservoir simulation) but is more complicated mathematically. One nonconvex example, the Buckley-Leverett equation, is discussed in the next chapter.

By far the most famous model problem in this field is **Burgers' equation**, in which $f(u) = \frac{1}{2}u^2$, so (3.13) becomes

$$u_t + uu_x = 0. \tag{3.14}$$

Actually this should be called the "inviscid Burgers' equation", since the equation studied by Burgers[5] also includes a viscous term:

$$u_t + uu_x = \epsilon u_{xx}. \tag{3.15}$$

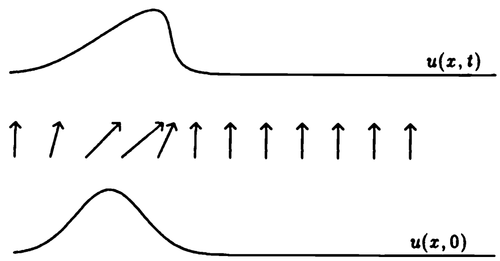

Figure 3.3. Characteristics and solution for Burgers' equation (small t).

This is about the simplest model that includes the nonlinear and viscous effects of fluid dynamics.

Around 1950, Hopf, and independently Cole, showed that the *exact* solution of the nonlinear equation (3.15) could be found using what is now called the **Cole-Hopf transformation**. This reduces (3.15) to a linear heat equation. See Chapter 4 of Whitham[97] for details.

Consider the inviscid equation (3.14) with smooth initial data. For small time, a solution can be constructed by following characteristics. Note that (3.14) looks like an advection equation, but with the advection velocity u equal to the value of the advected quantity. The characteristics satisfy

$$x'(t) = u(x(t), t) \tag{3.16}$$

and along each characteristic u is constant, since

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \frac{\partial}{\partial t}u(x(t),t) + \frac{\partial}{\partial x}u(x(t),t)\,x'(t) \\ &= u_t + uu_x \\ &= 0. \end{aligned} \tag{3.17}$$

Moreover, since u is constant on each characteristic, the slope $x'(t)$ is constant by (3.16) and so the characteristics are straight lines, determined by the initial data (see Fig. 3.3).

If the initial data is smooth, then this can be used to determine the solution $u(x,t)$ for small enough t that characteristics do not cross: For each (x,t) we can solve the equation

$$x = \xi + u(\xi, 0)t \tag{3.18}$$

for ξ and then

$$u(x,t) = u(\xi, 0). \tag{3.19}$$

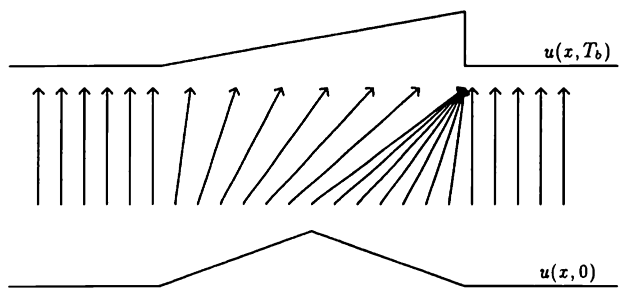

Figure 3.4. Shock formation in Burgers' equation.

3.3 Shock formation

For larger t the equation (3.18) may not have a unique solution. This happens when the characteristics cross, as will eventually happen if $u_x(x,0)$ is negative at any point. At the time T_b where the characteristics first cross, the function $u(x,t)$ has an infinite slope — the wave "breaks" and a shock forms. Beyond this point there is no classical solution of the PDE, and the weak solution we wish to determine becomes discontinuous.

Figure 3.4 shows an extreme example where the initial data is piecewise linear and many characteristics come together at once. More generally an infinite slope in the solution may appear first at just one point x, corresponding via (3.18) to the point ξ where the slope of the initial data is most negative. At this point the wave is said to "break", by analogy with waves on a beach.

EXERCISE 3.3. *Show that if we solve (3.14) with smooth initial data $u_0(x)$ for which $u_0'(x)$ is somewhere negative, then the wave will break at time*

$$T_b = \frac{-1}{\min u_0'(x)}. \tag{3.20}$$

Generalize this to arbitrary convex scalar equations.

For times $t > T_b$ some of the characteristics have crossed and so there are points x where there are three characteristics leading back to $t = 0$. One can view the "solution" u at such a time as a triple-valued function (see Fig. 3.5).

This sort of solution makes sense in some contexts, for example a breaking wave on a sloping beach can be modeled by the shallow water equations and, for a while at least, does behave as seen in Fig. 3.5, with fluid depth a triple-valued function.

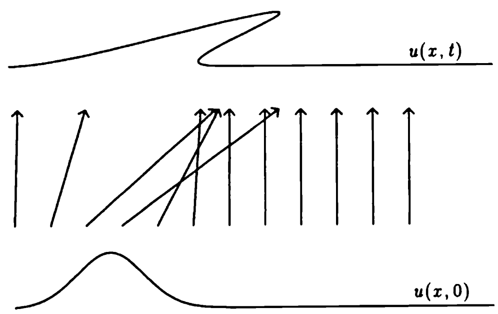

Figure 3.5. Triple-valued solution to Burgers' equation at time $t > T_b$.

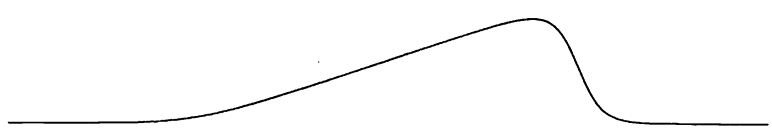

Figure 3.6. Solution to the viscous Burgers' equation at time T_b for the data shown in Figure 3.4.

However, in most physical situations this does not make sense. For example, the density of a gas cannot possibly be triple valued at a point. What happens instead at time T_b?

We can determine the correct physical behavior by adopting the vanishing viscosity approach. The equation (3.14) is a model of (3.15) valid only for small ϵ and smooth u. When it breaks down, we must return to (3.15). If the initial data is smooth and ϵ very small, then before the wave begins to break the ϵu_{xx} term is negligible compared to the other terms and the solutions to both PDEs look nearly identical. Figure 3.3, for example, would be essentially unchanged if we solved (3.15) with small ϵ rather than (3.14). However, as the wave begins to break, the second derivative term u_{xx} grows much faster than u_x, and at some point the ϵu_{xx} term is comparable to the other terms and begins to play a role. This term keeps the solution smooth for all time, preventing the breakdown of solutions that occurs for the hyperbolic problem.

For very small values of ϵ, the discontinuous solution at T_b shown in Figure 3.4 would

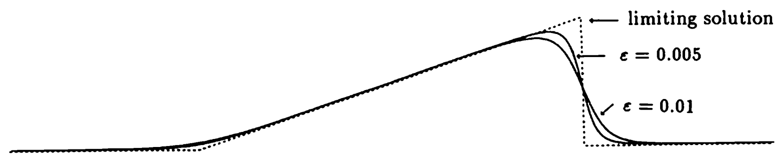

Figure 3.7. Solution to the viscous Burgers' equation for two different values of ϵ.

be replaced by a smooth continuous function as in Figure 3.6. As $\epsilon \to 0$ this becomes sharper and approaches the discontinuous solution of Figure 3.4.

For times $t > T_b$, such as was shown in Figure 3.5, the viscous solution for $\epsilon > 0$ would continue to be smooth and single valued, with a shape similar to that shown in Figure 3.6. The behavior as $\epsilon \to 0$ is indicated in Figure 3.7.

It is this vanishing viscosity solution that we hope to capture by solving the inviscid equation.

3.4 Weak solutions

A natural way to define a generalized solution of the inviscid equation that does not require differentiability is to go back to the integral form of the conservation law, and say that $u(x,t)$ is a generalized solution if (2.7) is satisfied for all $x_1,\ x_2,\ t_1,\ t_2$.

There is another approach that results in a different integral formulation that is often more convenient to work with. This is a mathematical technique that can be applied more generally to rewrite a differential equation in a form where less smoothness is required to define a "solution". The basic idea is to take the PDE, multiply by a smooth "test function", integrate one or more times over some domain, and then use integration by parts to move derivatives off the function u and onto the smooth test function. The result is an equation involving fewer derivatives on u, and hence requiring less smoothness.

In our case we will use test functions $\phi \in C_0^1(\mathbb{R}\times\mathbb{R})$. Here C_0^1 is the space of function that are continuously differentiable with "compact support". The latter requirement means that $\phi(x,t)$ is identically zero outside of some bounded set, and so the support of the function lies in a compact set.

If we multiply $u_t + f_x = 0$ by $\phi(x,t)$ and then integrate over space and time, we obtain

$$\int_0^\infty \int_{-\infty}^{+\infty} [\phi u_t + \phi f(u)_x]\, dx\, dt = 0. \tag{3.21}$$

Now integrate by parts, yielding

$$\int_0^\infty \int_{-\infty}^{+\infty} [\phi_t u + \phi_x f(u)]\, dx\, dt = -\int_{-\infty}^{\infty} \phi(x,0)u(x,0)\, dx. \tag{3.22}$$

Note that nearly all the boundary terms which normally arise through integration by parts drop out due to the requirement that ϕ have compact support, and hence vanishes at infinity. The remaining boundary term brings in the initial conditions of the PDE, which must still play a role in our weak formulation.

DEFINITION 3.1. *The function $u(x,t)$ is called a weak solution of the conservation law if (3.22) holds for all functions $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$.*

The advantage of this formulation over the original integral form (2.16) is that the integration in (3.22) is over a fixed domain, all of $\mathbb{R} \times \mathbb{R}^+$. The integral form (2.16) is over an arbitrary rectangle, and to check that $u(x,t)$ is a solution we must verify that this holds for all choices of x_1, x_2, t_1 and t_2. Of course, our new form (3.22) has a similar feature, we must check that it holds for all $\phi \in C_0^1$, but this turns out to be an easier task.

Mathematically the two integral forms are equivalent and we should rightly expect a more direct connection between the two that does not involve the differential equation. This can be achieved by considering special test functions $\phi(x,t)$ with the property that

$$\phi(x,t) = \begin{cases} 1 & \text{for } (x,t) \in [x_1, x_2] \times [t_1, t_2] \\ 0 & \text{for } (x,t) \notin [x_1 - \epsilon, x_2 + \epsilon] \times [t_1 - \epsilon, t_2 + \epsilon] \end{cases} \tag{3.23}$$

and with ϕ smooth in the intermediate strip of width ϵ. Then $\phi_x = \phi_t = 0$ except in this strip and so the integral (3.22) reduces to an integral over this strip. As $\epsilon \to 0$, ϕ_x and ϕ_t approach delta functions that sample u or $f(u)$ along the boundaries of the rectangle $[x_1, x_2] \times [t_1, t_2]$, so that (3.22) approaches the integral form (2.16). By making this rigorous, we can show that any weak solution satisfies the original integral conservation law.

The vanishing viscosity generalized solution we defined above is a weak solution in the sense of (3.22), and so this definition includes the solution we are looking for. Unfortunately, weak solutions are often not unique, and so an additional problem is often to identify *which* weak solution is the physically correct vanishing viscosity solution. Again, one would like to avoid working with the viscous equation directly, but it turns out that there are other conditions one can impose on weak solutions that are easier to check and will also pick out the correct solution. As noted in Chapter 1, these are usually called *entropy conditions* by analogy with the gas dynamics case. The vanishing viscosity solution is also called the **entropy solution** because of this.

3.5 The Riemann Problem

The conservation law together with piecewise constant data having a single discontinuity is known as the Riemann problem. As an example, consider Burgers' equation $u_t + uu_x = 0$

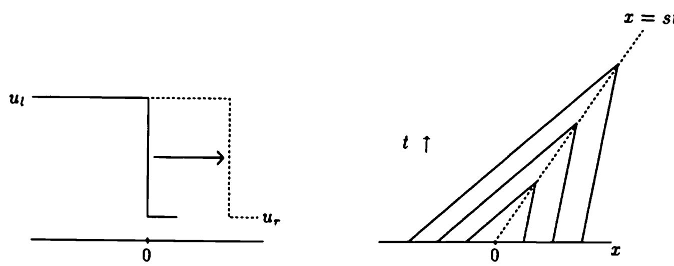

Figure 3.8. Shock wave.

with piecewise constant initial data

$$u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases} \tag{3.24}$$

The form of the solution depends on the relation between u_l and u_r.

Case I. $u_l > u_r$.

In this case there is a unique weak solution,

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases} \tag{3.25}$$

where

$$s = (u_l + u_r)/2 \tag{3.26}$$

is the **shock speed**, the speed at which the discontinuity travels. A general expression for the shock speed will be derived below. Note that characteristics in each of the regions where u is constant go *into* the shock (see Fig. 3.8) as time advances.

EXERCISE 3.4. *Verify that (3.25) is a weak solution to Burgers' equation by showing that (3.22) is satisfied for all* $\phi \in C_0^1$.

EXERCISE 3.5. *Show that the viscous equation (3.15) has a travelling wave solution of the form* $u^\epsilon(x,t) = w(x - st)$ *by deriving an ODE for* w *and verifying that this ODE has solutions of the form*

$$w(y) = u_r + \frac{1}{2}(u_l - u_r)\left[1 - \tanh((u_l - u_r)y/4\epsilon)\right] \tag{3.27}$$

with s *again given by (3.26). Note that* $w(y) \to u_l$ *as* $y \to -\infty$ *and* $w(y) \to u_r$ *as* $y \to +\infty$. *Sketch this solution and indicate how it varies as* $\epsilon \to 0$.

The smooth solution $u^\epsilon(x,t)$ found in Exercise 3.5 converges to the shock solution (3.25) as $\epsilon \to 0$, showing that our shock solution is the desired vanishing viscosity solution. The shape of $u^\epsilon(x,t)$ is often called the "viscous profile" for the shock wave.

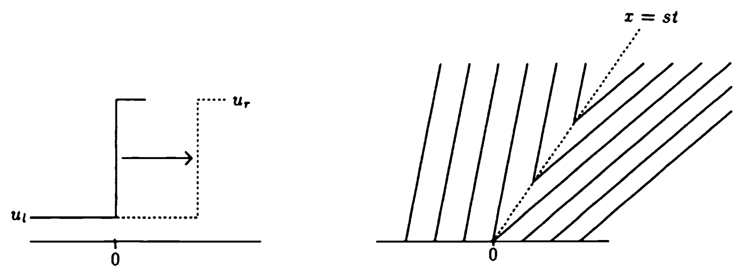

Figure 3.9. Entropy-violating shock.

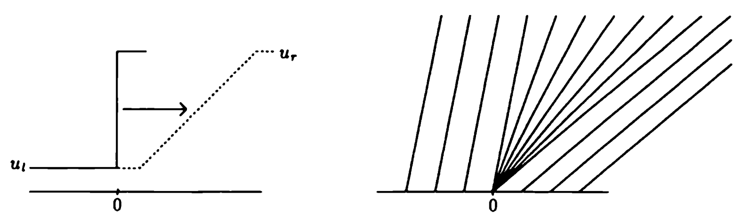

Figure 3.10. Rarefaction wave.

Case II. $u_l < u_r$.

In this case there are infinitely many weak solutions. One of these is again (3.25), (3.26) in which the discontinuity propagates with speed s. Note that characteristics now go *out* of the shock (Fig. 3.9) and that this solution is not stable to perturbations. If the data is smeared out slightly, or if a small amount of viscosity is added to the equation, the solution changes completely.

Another weak solution is the **rarefaction wave**

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t \le x \le u_r t \\ u_r & x > u_r t \end{cases} \tag{3.28}$$

This solution is stable to perturbations and is in fact the vanishing viscosity generalized solution (Fig. 3.10).

EXERCISE 3.6. *There are infinitely many other weak solutions to (3.14) when $u_l < u_r$. Show, for example, that*

$$u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \le x \le u_m t \\ x/t & u_m t \le x \le u_r t \\ u_r & x > u_r t \end{cases}$$

is a weak solution for any u_m with $u_l \leq u_m \leq u_r$ and $s_m = (u_l + u_m)/2$. Sketch the characteristics for this solution. Find a class of weak solutions with three discontinuities.

EXERCISE 3.7. *Show that for a general convex scalar problem (3.13) with data (3.24) and $u_l < u_r$, the rarefaction wave solution is given by*

$$u(x,t) = \begin{cases} u_l & x < f'(u_l)t \\ v(x/t) & f'(u_l)t \leq x \leq f'(u_r)t \\ u_r & x > f'(u_r)t \end{cases} \tag{3.29}$$

where $v(\xi)$ is the solution to $f'(v(\xi)) = \xi$.

3.6 Shock speed

The propagating shock solution (3.25) is a weak solution to Burgers' equation only if the speed of propagation is given by (3.26). The same discontinuity propagating at a different speed would not be a weak solution.

The speed of propagation can be determined by conservation. If M is large compared to st then by (2.15), $\int_{-M}^{M} u(x,t)\,dx$ must increase at the rate

$$\begin{aligned} \frac{d}{dt}\int_{-M}^{M} u(x,t)\,dx &= f(u_l) - f(u_r) \\ &= \frac{1}{2}(u_l + u_r)(u_l - u_r) \end{aligned} \tag{3.30}$$

for Burgers' equation. On the other hand, the solution (3.25) clearly has

$$\int_{-M}^{M} u(x,t)\,dx = (M + st)u_l + (M - st)u_r \tag{3.31}$$

so that

$$\frac{d}{dt}\int_{-M}^{M} u(x,t)\,dx = s(u_l - u_r). \tag{3.32}$$

Comparing (3.30) and (3.32) gives (3.26).

More generally, for arbitrary flux function $f(u)$ this same argument gives the following relation between the shock speed s and the states u_l and u_r, called the **Rankine-Hugoniot jump condition:**

$$f(u_l) - f(u_r) = s(u_l - u_r). \tag{3.33}$$

For scalar problems this gives simply

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{[f]}{[u]} \tag{3.34}$$

where $[\cdot]$ indicates the jump in some quantity across the discontinuity. Note that any jump is allowed, provided the speed is related via (3.34).

For systems of equations, $u_l - u_r$ and $f(u_r) - f(u_l)$ are both vectors while s is still a scalar. Now we cannot always solve for s to make (3.33) hold. Instead, only certain jumps $u_l - u_r$ are allowed, namely those for which the vectors $f(u_l) - f(u_r)$ and $u_l - u_r$ are linearly dependent.

EXAMPLE 3.1. For a linear system with $f(u) = Au$, (3.33) gives

$$A(u_l - u_r) = s(u_l - u_r), \tag{3.35}$$

i.e., $u_l - u_r$ must be an eigenvector of the matrix A and s is the associated eigenvalue. For a linear system, these eigenvalues are the characteristic speeds of the system. Thus discontinuities can propagate only along characteristics, a fact that we have already seen for the scalar case.

So far we have considered only piecewise constant initial data and shock solutions consisting of a single discontinuity propagating at constant speed. More typically, solutions have both smooth regions, where the PDEs are satisfied in the classical sense, and propagating discontinuities whose strength and speed vary as they interact with the smooth flow or collide with other shocks.

The Rankine-Hugoniot (R-H) conditions (3.33) hold more generally across any propagating shock, where now u_l and u_r denote the values immediately to the left and right of the discontinuity and s is the corresponding instantaneous speed, which varies along with u_l and u_r.

EXAMPLE 3.2. As an example, the following "N wave" is a solution to Burgers' equation:

$$u(x,t) = \begin{cases} x/t & -\sqrt{t} < x < \sqrt{t} \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

This solution has two shocks propagating with speeds $\pm\frac{1}{2\sqrt{t}}$. The right-going shock has left and right states $u_l = \sqrt{t}/t = 1/\sqrt{t}$, $u_r = 0$ and so the R-H condition is satisfied, and similarly for the left-going shock. See Figure 3.11.

To verify that the R-H condition must be instantaneously satisfied when u_l and u_r vary, we apply the same conservation argument as before but now to a small rectangle as shown in Figure 3.12, with $x_2 = x_1 + \Delta x$ and $t_2 = t_1 + \Delta t$. Assuming that u is smoothly varying on each side of the shock, and that the shock speed $s(t)$ is consequently also smoothly varying, we have the following relation between Δx and Δt:

$$\Delta x = s(t_1)\Delta t + O(\Delta t^2). \tag{3.37}$$

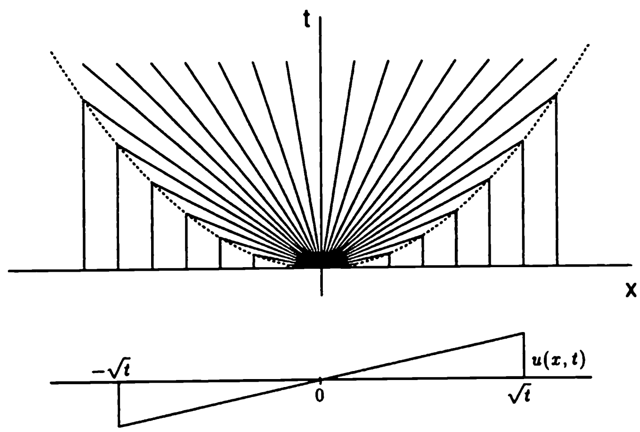

Figure 3.11. N wave solution to Burgers' equation.

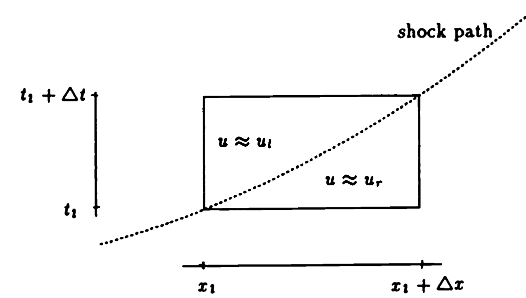

Figure 3.12. Region of integration for shock speed calculation.

From the integral form of the conservation law, we have

$$\int_{x_1}^{x_1+\Delta x} u(x, t_1 + \Delta t)\, dx = \int_{x_1}^{x_1+\Delta x} u(x, t_1)\, dx \qquad (3.38)$$
$$+ \int_{t_1}^{t_1+\Delta t} f(u(x_1, t))\, dt - \int_{t_1}^{t_1+\Delta t} f(u(x_1 + \Delta x, t))\, dt.$$

In the triangular portion of the infinitesimal rectangle that lies to the left of the shock, $u(x,t) = u_l(t_1) + O(\Delta t)$, while in the complementary triangle, $u(x,t) = u_r(t_1) + O(\Delta t)$. Using this in (3.38) gives

$$\Delta x\, u_l = \Delta x\, u_r + \Delta t\, f(u_l) - \Delta t\, f(u_r) + O(\Delta t^2).$$

Using the relation (3.37) in the above equation and then dividing by Δt gives

$$s\Delta t(u_l - u_r) = \Delta t(f(u_l) - f(u_r)) + O(\Delta t)$$

where s, u_l, and u_r are all evaluated at t_1. Letting $\Delta t \to 0$ gives the R-H condition (3.33).

EXERCISE 3.8. *Solve Burgers' equation with initial data*

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 < x < 2 \\ 0 & x > 2. \end{cases} \qquad (3.39)$$

Sketch the characteristics and shock paths in the x-t plane. Hint: The two shocks merge into one shock at some point.

The equal area rule. One technique that is useful for determining weak solutions by hand is to start with the "solution" constructed using characteristics (which may be multi-valued if characteristics cross) and then eliminate the multi-valued parts by inserting shocks. The shock location can be determined by the "equal area rule", which is best understood by looking at Figure 3.13. The shock is located such that the shaded regions cut off on either side have equal areas, as in Figure 3.13b. This is a consequence of conservation — the integral of the discontinuous weak solution (shaded area in Figure 3.13c) must be the same as the area "under" the multi-valued solution (shaded area in 3.13a), since both "solve" the same conservation law.

3.7 Manipulating conservation laws

One danger to observe in dealing with conservation laws is that transforming the differential form into what appears to be an equivalent differential equation may not give an equivalent equation in the context of weak solutions.

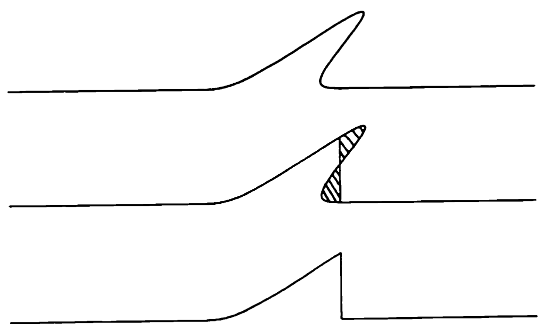

Figure 3.13. Equal area rule for shock location.

EXAMPLE 3.3. If we multiply Burgers' equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \tag{3.40}$$

by $2u$, we obtain $2uu_t + 2u^2u_x = 0$, which can be rewritten as

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x = 0. \tag{3.41}$$

This is again a conservation law, now for u^2 rather than u itself, with flux function $f(u^2) = \frac{2}{3}(u^2)^{3/2}$. The differential equations (3.40) and (3.41) have precisely the same smooth solutions. However, they have different weak solutions, as we can see by considering the Riemann problem with $u_l > u_r$. The unique weak solution of (3.40) is a shock traveling at speed

$$s_1 = \frac{\left[\frac{1}{2}u^2\right]}{[u]} = \frac{1}{2}(u_l + u_r), \tag{3.42}$$

whereas the unique weak solution to (3.41) is a shock traveling at speed

$$s_2 = \frac{\left[\frac{2}{3}u^3\right]}{[u^2]} = \frac{2}{3}\left(\frac{u_r^3 - u_l^3}{u_r^2 - u_l^2}\right). \tag{3.43}$$

It is easy to check that

$$s_2 - s_1 = \frac{1}{6}\frac{(u_l - u_r)^2}{(u_l + u_r)} \tag{3.44}$$

and so $s_2 \neq s_1$ when $u_l \neq u_r$, and the two equations have different weak solutions. The derivation of (3.41) from (3.40) requires manipulating derivatives in a manner that is valid only when u is smooth.

3.8 Entropy conditions

As demonstrated above, there are situations in which the weak solution is not unique and an additional condition is required to pick out the physically relevant vanishing viscosity solution. The condition which defines this solution is that it should be the limiting solution of the viscous equation as $\epsilon \to 0$, but this is not easy to work with. We want to find simpler conditions.

For scalar equations there is an obvious condition suggested by Figures 3.8 and 3.10. A shock should have characteristics going into the shock, as time advances. A propagating discontinuity with characteristics coming *out* of it, as in Figure 3.9, is unstable to perturbations. Either smearing out the initial profile a little, or adding some viscosity to the system, will cause this to be replaced by a rarefaction fan of characteristics, as in Figure 3.10. This gives our first version of the entropy condition:

ENTROPY CONDITION (VERSION I): *A discontinuity propagating with speed s given by (3.33) satisfies the entropy condition if*

$$f'(u_l) > s > f'(u_r). \tag{3.45}$$

Note that $f'(u)$ is the characteristic speed. For convex f, the Rankine-Hugoniot speed s from (3.34) must lie between $f'(u_l)$ and $f'(u_r)$, so (3.45) reduces to simply the requirement that $f'(u_l) > f'(u_r)$, which again by convexity requires $u_l > u_r$.

A more general form of this condition, due to Oleinik, applies also to nonconvex scalar flux functions f:

ENTROPY CONDITION (VERSION II): *$u(x,t)$ is the entropy solution if all discontinuities have the property that*

$$\frac{f(u)-f(u_l)}{u-u_l} \geq s \geq \frac{f(u)-f(u_r)}{u-u_r} \tag{3.46}$$

for all u between u_l and u_r.

For convex f, this requirement reduces to (3.45).

Another form of the entropy condition is based on the spreading of characteristics in a rarefaction fan. If $u(x,t)$ is an increasing function of x in some region, then characteristics spread out if $f'' > 0$. The rate of spreading can be quantified, and gives the following condition, also due to Oleinik[57].

ENTROPY CONDITION (VERSION III): *$u(x,t)$ is the entropy solution if there is a constant $E > 0$ such that for all $a > 0$, $t > 0$ and $x \in \mathbb{R}$,*

$$\frac{u(x+a,t)-u(x,t)}{a} < \frac{E}{t}. \tag{3.47}$$

Note that for a discontinuity propagating with constant left and right states u_l and u_r, this can be satisfied only if $u_r - u_l \leq 0$, so this agrees with (3.45). The form of (3.47) may seem unnecessarily complicated, but it turns out to be easier to apply in some contexts. In particular, this formulation has advantages in studying numerical methods. One problem we face in developing numerical methods is guaranteeing that they converge to the correct solution. Some numerical methods are known to converge to the *wrong* weak solution in some instances. The criterion (3.45) is hard to apply to discrete solutions — a discrete approximation defined only at grid points is in some sense discontinuous everywhere. If $U_j < U_{j+1}$ at some grid point, how do we determine whether this is a jump that violates the entropy condition, or is merely part of a smooth approximation of a rarefaction wave? Intuitively, we know the answer: it's part of a smooth approximation, and therefore acceptable, if the size of this jump is $O(\Delta x)$ as we refine the grid and $\Delta x \to 0$. To turn this into a proof that some method converges to the correct solution, we must quantify this requirement and (3.47) gives what we need. Taking $a = \Delta x$, we must ensure that there is a constant $E > 0$ such that

$$U_{j+1}(t) - U_j(t) < \left(\frac{E}{t}\right)\Delta x \tag{3.48}$$

for all $t > 0$ as $\Delta x \to 0$. This inequality can often be established for discrete methods.

In fact, Oleinik's original proof that an entropy solution to (3.13) satisfying (3.47) always exists proceeds by defining such a discrete approximation and then taking limits. This is also presented in Theorem 16.1 of Smoller[77].

3.8.1 Entropy functions

Yet another approach to the entropy condition is to define an entropy function $\eta(u)$ for which an additional conservation law holds for smooth solutions that becomes an inequality for discontinuous solutions. In gas dynamics, there exists a physical quantity called the entropy that is known to be constant along particle paths in smooth flow and to jump to a higher value as the gas crosses a shock. It can never jump to a lower value, and this gives the physical entropy condition that picks out the correct weak solution in gas dynamics.

Suppose some function $\eta(u)$ satisfies a conservation law of the form

$$\eta(u)_t + \psi(u)_x = 0 \tag{3.49}$$

for some **entropy flux** $\psi(u)$. Then we can obtain from this, for smooth u,

$$\eta'(u)u_t + \psi'(u)u_x = 0. \tag{3.50}$$

Recall that the conservation law (3.13) can be written as $u_t + f'(u)u_x = 0$. Multiply this by $\eta'(u)$ and compare with (3.50) to obtain

$$\psi'(u) = \eta'(u)f'(u). \tag{3.51}$$

For a scalar conservation law this equation admits many solutions $\eta(u)$, $\psi(u)$. For a system of equations η and ψ are still *scalar* functions, but now (3.51) reads $\nabla\psi(u) = f'(u)\nabla\eta(u)$, which is a system of m equations for the two variables η and ψ. If $m > 2$ this may have no solutions.

An additional condition we place on the entropy function is that it be *convex*, $\eta''(u) > 0$, for reasons that will be seen below.

The entropy $\eta(u)$ is conserved for *smooth* flows by its definition. For discontinuous solutions, however, the manipulations performed above are not valid. Since we are particularly interested in how the entropy behaves for the vanishing viscosity weak solution, we look at the related viscous problem and will then let the viscosity tend to zero. The viscous equation is

$$u_t + f(u)_x = \epsilon u_{xx}. \tag{3.52}$$

Since solutions to this equation are always smooth, we can derive the corresponding evolution equation for the entropy following the same manipulations we used for smooth solutions of the inviscid equation, multiplying (3.52) by $\eta'(u)$ to obtain

$$\eta(u)_t + \psi(u)_x = \epsilon\eta'(u)u_{xx}. \tag{3.53}$$

We can now rewrite the right hand side to obtain

$$\eta(u)_t + \psi(u)_x = \epsilon(\eta'(u)u_x)_x - \epsilon\eta''(u)u_x^2. \tag{3.54}$$

Integrating this equation over the rectangle $[x_1, x_2] \times [t_1, t_2]$ gives

$$\begin{aligned} \int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx\, dt &= \epsilon\int_{t_1}^{t_2} [\eta'(u(x_2,t))u_x(x_2,t) - \eta'(u(x_1,t))u_x(x_1,t)]\, dt \\ &\quad -\epsilon\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta''(u)u_x^2\, dx\, dt. \end{aligned}$$

As $\epsilon \to 0$, the first term on the right hand side vanishes. (This is clearly true if u is smooth at x_1 and x_2, and can be shown more generally.) The other term, however, involves integrating u_x^2 over the $[x_1, x_2] \times [t_1, t_2]$. If the limiting weak solution is discontinuous along a curve in this rectangle, then this term will not vanish in the limit. However, since $\epsilon > 0$, $u_x^2 > 0$ and $\eta'' > 0$ (by our convexity assumption), we can conclude that the right hand side is nonpositive in the limit and hence the vanishing viscosity weak solution satisfies

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \eta(u)_t + \psi(u)_x \, dx\, dt \le 0 \tag{3.55}$$

for all x_1, x_2, t_1 and t_2. Alternatively, in integral form,

$$\int_{x_1}^{x_2} \eta(u(x,t))\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \psi(u(x,t))\,dt\Big|_{x_1}^{x_2} \le 0, \tag{3.56}$$

i.e.,

$$\begin{aligned}\int_{x_1}^{x_2} \eta(u(x,t_2))\,dx \;\le\; & \int_{x_1}^{x_2} \eta(u(x,t_1))\,dx \\ & - \left(\int_{t_1}^{t_2} \psi(u(x_2,t))\,dt - \int_{t_1}^{t_2} \psi(u(x_1,t))\,dt\right).\end{aligned} \tag{3.57}$$

Consequently, the total integral of η is not necessarily conserved, but can only *decrease*. (Note that our mathematical assumption of convexity leads to an "entropy function" that decreases, whereas the physical entropy in gas dynamics increases.) The fact that (3.55) holds for all x_1, x_2, t_1 and t_2 is summarized by saying that $\eta(u)_t + \psi(u)_x \le 0$ in the weak sense. This gives our final form of the entropy condition, called the **entropy inequality**.

ENTROPY CONDITION (VERSION IV): *The function $u(x,t)$ is the entropy solution of (3.13) if, for all convex entropy functions and corresponding entropy fluxes, the inequality*

$$\eta(u)_t + \psi(u)_x \le 0 \tag{3.58}$$

is satisfied in the weak sense.

This formulation is also useful in analyzing numerical methods. If a discrete form of this entropy inequality is known to hold for some numerical method, then it can be shown that the method converges to the entropy solution.

Just as for the conservation law, an alternative weak form of the entropy condition can be formulated by integrating against smooth test functions ϕ, now required to be nonnegative since the entropy condition involves an inequality. The **weak form of the entropy inequality** is

$$\begin{aligned}&\int_0^\infty \int_{-\infty}^\infty \phi_t(x,t)\eta(u(x,t)) + \phi_x(x,t)\psi(u(x,t))\,dx\,dt \\ &\qquad \le -\int_{-\infty}^\infty \phi(x,0)\eta(u(x,0))\,dx\end{aligned} \tag{3.59}$$

for all $\phi \in C_0^1(\mathbb{R}\times\mathbb{R})$ with $\phi(x,t) \ge 0$ for all x, t.

EXAMPLE 3.4. Consider Burgers' equation with $f(u) = \frac{1}{2}u^2$ and take $\eta(u) = u^2$. Then (3.51) gives $\psi'(u) = 2u^2$ and hence $\psi(u) = \frac{2}{3}u^3$. Then entropy condition (3.58) reads

$$(u^2)_t + \left(\frac{2}{3}u^3\right)_x \le 0. \tag{3.60}$$

For smooth solutions of Burgers' equation this should hold with equality, as we have already seen in Example 3.3. If a discontinuity is present, then integrating $(u^2)_t + (\frac{2}{3}u^3)_x$ over an infinitesmal rectangle as in Figure 3.12 gives

$$\begin{aligned}\int_{x_1}^{x_2} u^2(x,t)\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{2}{3}u^3(x,t)\,dt\Big|_{x_1}^{x_2} &= s_1\Delta t(u_l^2 - u_r^2) + \frac{2}{3}\Delta t(u_r^3 - u_l^3) + O(\Delta t^2) \\ &= \Delta t(u_l^2 - u_r^2)(s_1 - s_2) + O(\Delta t^2) \\ &= -\frac{1}{6}(u_l - u_r)^3\Delta t + O(\Delta t^2)\end{aligned}$$

where s_1 and s_2 are given by (3.42) and (3.43) and we have used (3.44). For small $\Delta t > 0$, the $O(\Delta t^2)$ term will not affect the sign of this quantity and so the weak form (3.56) is satisfied if and only if $(u_l - u_r)^3 > 0$, and hence the only allowable discontinuities have $u_l > u_r$, as expected.