Journal of Computational and Applied Mathematics 255 (2014) 616–634

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Generalized Hermite spectral method matching asymptotic behaviors

Chao Zhang [a], Ben-yu Guo [b,c,d,*]

[a] Jiangsu Normal University, Xuzhou, 221116, China
[b] Shanghai Normal University, Shanghai, 200234, China
[c] Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, 200234, China
[d] Division of Computational Science of E-institute of Shanghai Universities, Shanghai, 200234, China

ARTICLE INFO

Article history:
Received 6 September 2012
Received in revised form 13 June 2013

MSC:
41A30
65L60
65M70
34B40
35L70

Keywords:
Generalized Hermite orthogonal approximation and interpolation
Spectral method on the whole line

ABSTRACT

In this paper, we propose the generalized Hermite spectral method by using a family of new generalized Hermite functions, which are mutually orthogonal with the weight function $(1 + x^2)^{-\gamma}$, γ being an arbitrary real number. We establish the basic results on the corresponding orthogonal approximation and interpolation, which simulate the asymptotic behaviors of approximated functions at infinity reasonably. As examples of applications, the spectral schemes are provided for two model problems. Numerical results demonstrate their spectral accuracy in space.

1. Introduction

During the past two decades, more and more attentions were paid to numerical solutions of differential equations defined on unbounded domains. For solving problems defined on the whole line and the related unbounded domains, we may use the Hermite orthogonal approximation and the Hermite–Gauss interpolation. Guo [1], and Guo and Xu [2] developed the spectral and pseudospectral methods for nonlinear partial differential equations, by using the standard Hermite polynomials which are mutually orthogonal with the weight function e^{-x^2}. Weideman [3] presented the related implementations. These methods are also available, even if the approximated solutions grow like $e^{\alpha x^2}(\alpha < \frac{1}{2})$ as $|x|$ increases. However, the small global numerical errors with the weight function e^{-x^2} do not imply the small point-wise numerical errors for large $|x|$ automatically. Meanwhile, Funaro and Kavian [4] considered the spectral method for linear parabolic equations by using the orthogonal system with the weight function $e^{\gamma x^2}(\gamma > 0)$. Fok, Guo and Tang [5] applied a similar approach coupled with finite difference approximation, to the simplified Fokker–Planck equation. Such methods are only suitable for problems with solutions behaving like $e^{-\alpha x^2}(\alpha > \frac{1}{2}\gamma)$ at infinity. On the other hand, Guo, Shen and Xu [6] provided the spectral and pseudospectral methods for the Dirac equation with the solution behaving like $(1 + x^2)^{-\frac{1}{2}\alpha}(\alpha > 1)$ at infinity, by using the Hermite functions which are mutually orthogonal with the weight function 1. We also refer the readers to the work of

* Corresponding author at: Shanghai Normal University, Shanghai, 200234, China. Tel.: +86 21 64322851; fax: +86 21 64323364.
E-mail addresses: zcxz1977@163.com (C. Zhang), byguo@shnu.edu.cn (B.Y. Guo).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.06.018

Boyd [7,8], Ma, Sun and Tang [9], Ma and Zhao [10], and Xiang and Wang [11]. In many practical problems, the solutions might behave like $(1+x^2)^{\frac{1}{2}\alpha}$ for large $|x|$, α being certain real number. In these cases, it seems reasonable to adopt the orthogonal approximation with the weight function like $(1+x^2)^{-\gamma}$, $\gamma > \alpha + \frac{1}{2}$.

In this paper, we propose a family of new generalized Hermite functions, which are mutually orthogonal with the weight function $(1+x^2)^{-\gamma}$, γ being any real number. We establish the basic results on the corresponding Hermite orthogonal approximation and Hermite–Gauss interpolation. By adjusting the parameter γ suitably, such approximations might simulate the asymptotic behaviors of approximated functions at infinity reasonably, and so play an important role in the related Hermite spectral and pseudospectral methods for differential equations with various asymptotic behaviors at infinity. As examples of applications, we provide the spectral schemes for a linear model problem and the sine–Gordon equation, and prove their spectral accuracy in space. The numerical results demonstrate the effectiveness of the suggested algorithms.

This paper is organized as follows. The next section is for preliminaries. In Section 3, we introduce the new generalized Hermite orthogonal approximation and Hermite–Gauss interpolation. In Section 4, we propose the spectral schemes for two model problems, and present some numerical results. The final section is for concluding remarks.

2. Preliminaries

In this section, we recall some results on the existing Hermite orthogonal approximation and Hermite–Gauss interpolation.

Let $\Lambda = \{x \mid -\infty < x < \infty\}$ and $\chi(x)$ be a certain weight function. For any integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(\cdot,\cdot)_{r,\chi,\Lambda}$, the semi-norm $|\cdot|_{r,\chi,\Lambda}$ and the norm $\|\cdot\|_{r,\chi,\Lambda}$. In particular, the inner product and the norm of $L^2_\chi(\Lambda)$ are denoted by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$, respectively. We omit the subscript χ in notations whenever $\chi(x) \equiv 1$. For simplicity of statements, we denote $\frac{d^k v}{dx^k}$ by $\partial_x^k v$, etc.

Let $H_l(x)$ be the standard Hermite polynomial of degree l. For any $\beta > 0$, the scaled Hermite functions are given by

$$H_l^\beta(x) = \frac{1}{\sqrt{2^l l!}} e^{-\frac{1}{2}\beta^2 x^2} H_l(\beta x), \quad l \geq 0.$$

They are the eigenfunctions of the following singular Sturm–Liouville problem,

$$e^{\frac{1}{2}\beta^2 x^2}\partial_x\left(e^{-\beta^2 x^2}\partial_x\left(e^{\frac{1}{2}\beta^2 x^2} v(x)\right)\right) + \lambda_l^\beta v(x) = 0, \quad \lambda_l^\beta = 2\beta^2 l,\ l \geq 0. \tag{2.1}$$

Let $\delta_{l,m}$ be the Kronecker symbol. The set of all $H_l^\beta(x)$ is a complete $L^2(\Lambda)$-orthogonal system, i.e.,

$$\int_\Lambda H_l^\beta(x) H_m^\beta(x) dx = \frac{\sqrt{\pi}}{\beta}\delta_{l,m}. \tag{2.2}$$

For any $v \in L^2(\Lambda)$, we have

$$v(x) = \sum_{l=0}^\infty v_l^\beta H_l^\beta(x), \tag{2.3}$$

with

$$v_l^\beta = \frac{\beta}{\sqrt{\pi}} \int_\Lambda v(x) H_l^\beta(x) dx.$$

Let

$$\mathcal{Q}_N^\beta(\Lambda) = \mathrm{span}\{H_l^\beta(x),\ 0 \leq l \leq N\}.$$

The $L^2(\Lambda)$-orthogonal projection $P_{N,\beta,\Lambda} : L^2(\Lambda) \to \mathcal{Q}_N^\beta(\Lambda)$ is defined by

$$(P_{N,\beta,\Lambda} v - v, \phi) = 0, \quad \forall \phi \in \mathcal{Q}_N^\beta(\Lambda). \tag{2.4}$$

For any integer $r \geq 0$, we define the space

$$H^r_{A,\beta}(\Lambda) = \{u \mid \|u\|_{H^r_{A,\beta}(\Lambda)} < \infty\},$$

equipped with the norm

$$\|u\|_{H^r_{A,\beta}(\Lambda)} = \left(\sum_{k=0}^r \left\|(\beta^4 x^2 + \beta^2)^{\frac{r-k}{2}} \partial_x^k u\right\|_\Lambda^2\right)^{\frac{1}{2}}.$$

Throughout this paper, we denote by c a generic positive constant independent of any function, N and β. According to Theorem 2.1 of [11], we have the following result.

Lemma 2.1. *If $v \in H^r_{A,\beta}(\Lambda)$ and integers $0 \le k \le r$, then*

$$\|P_{N,\beta,\Lambda}v - v\|_{k,\Lambda} \le c(\beta^2 N)^{\frac{k-r}{2}} \|v\|_{H^r_{A,\beta}(\Lambda)}. \tag{2.5}$$

The above result with $\beta = 1$ was first given by Guo, Shen and Xu; see Theorem 2.3 of [6].

We now turn to the related Hermite–Gauss interpolation. Let $\sigma_{N,j}$ and $\omega_{N,j}$ be the nodes and the weights of the standard Hermite–Gauss interpolation, $0 \le j \le N$ (cf. [2]). We take the nodes and the weights of the scaled Hermite–Gauss interpolation as follows,

$$\sigma^\beta_{N,j} = \frac{\sigma_{N,j}}{\beta}, \qquad \omega^\beta_{N,j} = \frac{1}{\beta}\omega_{N,j} e^{\sigma^2_{N,j}}, \quad 0 \le j \le N.$$

Accordingly, we introduce the following discrete inner product and norm,

$$(u, v)_{N,\beta,\Lambda} = \sum_{j=0}^{N} u(\sigma^\beta_{N,j}) v(\sigma^\beta_{N,j}) \omega^\beta_{N,j}, \qquad \|v\|_{N,\beta,\Lambda} = (v, v)^{\frac{1}{2}}_{N,\beta,\Lambda}.$$

For any $\phi \in \mathcal{Q}^\beta_m(\Lambda)$ and $\psi \in \mathcal{Q}^\beta_{2N+1-m}(\Lambda)$, there exist the following polynomials,

$$\phi^\beta(x) = e^{\frac{1}{2}x^2} \phi\left(\frac{x}{\beta}\right), \qquad \psi^\beta(x) = e^{\frac{1}{2}x^2} \psi\left(\frac{x}{\beta}\right).$$

With the aid of exactness of the standard Hermite–Gauss numerical quadrature, a careful calculation shows that

$$\begin{aligned}
(\phi, \psi)_{N,\beta,\Lambda} &= \sum_{j=0}^{N} \phi(\sigma^\beta_{N,j}) \psi(\sigma^\beta_{N,j}) \omega^\beta_{N,j} \\
&= \frac{1}{\beta} \sum_{j=0}^{N} \phi^\beta(\sigma_{N,j}) \psi^\beta(\sigma_{N,j}) \omega_{N,j} \\
&= \frac{1}{\beta} \int_\Lambda \phi^\beta(x) \psi^\beta(x) e^{-x^2} dx \\
&= \frac{1}{\beta} \int_\Lambda \phi\left(\frac{x}{\beta}\right) \psi\left(\frac{x}{\beta}\right) dx = (\phi, \psi)_\Lambda, \quad \forall \phi \in \mathcal{Q}^\beta_m(\Lambda),\ \psi \in \mathcal{Q}^\beta_{2N+1-m}(\Lambda).
\end{aligned} \tag{2.6}$$

For any $v \in C(\Lambda)$, the corresponding Hermite–Gauss interpolation $I_{N,\beta,\Lambda}v \in \mathcal{Q}^\beta_N(\Lambda)$ is determined uniquely by

$$I_{N,\beta,\Lambda}v(\sigma^\beta_{N,j}) = v(\sigma^\beta_{N,j}), \quad 0 \le j \le N. \tag{2.7}$$

For describing the approximation error of $I_{N,\beta,\Lambda}v$, we need some preparation. First, according to Lemma 2.1 of [11], we know that for any $\phi \in \mathcal{Q}^\beta_N(\Lambda)$,

$$\|\partial^k_x \phi\|_\Lambda \le c\beta^k N^{\frac{k}{2}} \|\phi\|_\Lambda. \tag{2.8}$$

Next, we can follow the same line as the proof of Lemma 3.1 of [6] to verify that for any $v \in H^1(\Lambda)$,

$$\|v\|_{N,\beta,\Lambda} = \|I_{N,\beta,\Lambda}v\|_{N,\beta,\Lambda} \le c\left(\|v\|_\Lambda + N^{-\frac{1}{6}} \|\partial_x v\|_\Lambda\right). \tag{2.9}$$

Lemma 2.2. *If $v \in H^r_{A,\beta}(\Lambda)$, integers $r \ge 1$ and $0 \le k \le r$, then*

$$\|I_{N,\beta,\Lambda}v - v\|_{k,\Lambda} \le c(\beta^k + 1)(\beta^2 N)^{\frac{1}{3} + \frac{k-r}{2}} \|v\|_{H^r_{A,\beta}(\Lambda)}. \tag{2.10}$$

Proof. Clearly, $I_{N,\beta,\Lambda}P_{N,\beta,\Lambda}v = P_{N,\beta,\Lambda}v$. By using (2.8), (2.9) and Lemma 2.1 successively, we verify that

$$\begin{aligned}
\|I_{N,\beta,\Lambda}v - P_{N,\beta,\Lambda}v\|_{k,\Lambda} &\le c\beta^k N^{\frac{k}{2}} \|I_{N,\beta,\Lambda}(P_{N,\beta,\Lambda}v - v)\|_\Lambda \\
&\le c\beta^k N^{\frac{k}{2}} \left(\|P_{N,\beta,\Lambda}v - v\|_\Lambda + N^{-\frac{1}{6}} \|\partial_x(P_{N,\beta,\Lambda}v - v)\|_\Lambda\right) \\
&\le c\beta^k (\beta^2 N)^{\frac{1}{3} + \frac{k-r}{2}} \|v\|_{H^r_{A,\beta}(\Lambda)}.
\end{aligned}$$

Using Lemma 2.1 again yields

$$\begin{aligned}\|I_{N,\beta,\Lambda}v - v\|_{k,\Lambda} &\le \|P_{N,\beta,\Lambda}v - v\|_{k,\Lambda} + \|I_{N,\beta,\Lambda}v - P_{N,\beta,\Lambda}v\|_{k,\Lambda}\\ &\le c(\beta^k + 1)(\beta^2 N)^{\frac{1}{3}+\frac{k-r}{2}}\|v\|_{H^r_{A,\beta}(\Lambda)}.\end{aligned}$$

This ends the proof. □

Furthermore, thanks to (2.7), (2.6) and (2.10), we deduce that

$$\begin{aligned}|(v,\phi)_\Lambda - (v,\phi)_{N,\beta,\Lambda}| &= |(v - I_{N,\beta,\Lambda}v, \phi)_\Lambda|\\ &\le c\|v - I_{N,\beta,\Lambda}v\|_\Lambda \|\phi\|_\Lambda\\ &\le c(\beta^2 N)^{\frac{1}{3}-\frac{r}{2}}\|v\|_{H^r_{A,\beta}(\Lambda)}\|\phi\|_\Lambda. \end{aligned} \tag{2.11}$$

3. New generalized Hermite orthogonal approximation and Hermite–Gauss interpolation

In this section, we propose the new generalized Hermite approximation.

3.1. New generalized Hermite functions

For any real number γ, the new generalized Hermite functions are defined by

$$\hat{H}_l^{\beta,\gamma}(x) = (1+x^2)^{\frac{\gamma}{2}} H_l^\beta(x), \quad \beta > 0,\ l \ge 0.$$

Due to (2.1), $\hat{H}_l^{\beta,\gamma}(x)$ is the l-th eigenfunction of the following Sturm–Liouville problem,

$$e^{\frac{1}{2}\beta^2x^2}\partial_x\left(e^{-\beta^2x^2}\partial_x\left((1+x^2)^{-\frac{\gamma}{2}}e^{\frac{1}{2}\beta^2x^2}v(x)\right)\right) + \lambda_l^\beta(1+x^2)^{-\frac{\gamma}{2}}v(x) = 0, \quad l \ge 0. \tag{3.1}$$

The weight function $\omega_\gamma(x) = (1+x^2)^{-\gamma}$. By virtue of (2.2), the set of all $\hat{H}_l^{\beta,\gamma}(x)$ is a complete $L^2_{\omega_\gamma}(\Lambda)$-orthogonal system, namely,

$$\int_\Lambda \hat{H}_l^{\beta,\gamma}(x)\hat{H}_m^{\beta,\gamma}(x)\omega_\gamma(x)dx = \frac{\sqrt{\pi}}{\beta}\delta_{l,m}. \tag{3.2}$$

Thus, for any $v \in L^2_{\omega_\gamma}(\Lambda)$, we have

$$v(x) = \sum_{l=0}^{\infty}\hat{v}_l^{\beta,\gamma}\hat{H}_l^{\beta,\gamma}(x), \tag{3.3}$$

with

$$\hat{v}_l^{\beta,\gamma} = \frac{\beta}{\sqrt{\pi}}\int_\Lambda v(x)\hat{H}_l^{\beta,\gamma}(x)\omega_\gamma(x)dx. \tag{3.4}$$

3.2. Generalized Hermite orthogonal approximation

Let N be any positive integer, and

$$\hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda) = \mathrm{span}\{\hat{H}_l^{\beta,\gamma}(x), 0 \le l \le N\}.$$

The $L^2_{\omega_\gamma}(\Lambda)$-orthogonal projection $\hat{P}_{N,\beta,\gamma,\Lambda} : L^2_{\omega_\gamma}(\Lambda) \to \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda)$ is defined by

$$(\hat{P}_{N,\beta,\gamma,\Lambda}v - v, \phi)_{\omega_\gamma,\Lambda} = 0, \quad \forall \phi \in \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda),$$

or equivalently,

$$\hat{P}_{N,\beta,\gamma,\Lambda}v(x) = \sum_{l=0}^{N}\hat{v}_l^{\beta,\gamma}\hat{H}_l^{\beta,\gamma}(x). \tag{3.5}$$

In order to estimate the approximation error, we introduce the following Sturm–Liouville operator,

$$\mathcal{A}_{\beta,\gamma}v(x) = -(1+x^2)^{\frac{\gamma}{2}}e^{\frac{1}{2}\beta^2x^2}\partial_x\left(e^{-\beta^2x^2}\partial_x\left((1+x^2)^{-\frac{\gamma}{2}}e^{\frac{1}{2}\beta^2x^2}v(x)\right)\right).$$

According to (3.1), we have

$$\mathcal{A}_{\beta,\gamma}\hat{H}_l^{\beta,\gamma}(x)=\lambda_l^{\beta}\hat{H}_l^{\beta,\gamma}(x),\quad l\geq 0. \tag{3.6}$$

Using integration by parts, we obtain

$$\begin{aligned}(v,\hat{H}_l^{\beta,\gamma})_{\omega_\gamma,\Lambda}&=(\lambda_l^{\beta})^{-1}(v,\mathcal{A}_{\beta,\gamma}\hat{H}_l^{\beta,\gamma})_{\omega_\gamma,\Lambda}\\&=(\lambda_l^{\beta})^{-1}(\mathcal{A}_{\beta,\gamma}v,\hat{H}_l^{\beta,\gamma})_{\omega_\gamma,\Lambda}.\end{aligned} \tag{3.7}$$

Therefore, if u, v are in the domain of the operator $\mathcal{A}_{\beta,\gamma}$, then

$$(\mathcal{A}_{\beta,\gamma}u,v)_{\omega_\gamma,\Lambda}=(u,\mathcal{A}_{\beta,\gamma}v)_{\omega_\gamma,\Lambda}.$$

Hence, $\mathcal{A}_{\beta,\gamma}$ is a positive definite and self-conjugate operator. Thus, we could define the following spaces with any integer $r\geq 0$,

$$D(\mathcal{A}^r_{\beta,\gamma})=\{v\mid \mathcal{A}^k_{\beta,\gamma}v\in L^2_{\omega_\gamma}(\Lambda),\ 0\leq k\leq r\},$$

equipped with the following semi-norm and norm,

$$|v|_{D(\mathcal{A}^r_{\beta,\gamma})}=\|\mathcal{A}^r_{\beta,\gamma}v\|_{\omega_\gamma,\Lambda},\qquad \|v\|_{D(\mathcal{A}^r_{\beta,\gamma})}=\left(\sum_{k=0}^{r}|v|^2_{D(\mathcal{A}^k_{\beta,\gamma})}\right)^{\frac{1}{2}}.$$

Theorem 3.1. *If $v\in D(\mathcal{A}^r_{\beta,\gamma})$ and integers $0\leq k\leq r$, then*

$$|\hat{P}_{N,\beta,\gamma,\Lambda}v-v|_{D(\mathcal{A}^k_{\beta,\gamma})}\leq c(\beta^2N)^{k-r}|v|_{D(\mathcal{A}^r_{\beta,\gamma})}. \tag{3.8}$$

Proof. We use (3.3), (3.5), (3.6), (3.2) and (3.4) successively, to deduce that

$$\begin{aligned}|\hat{P}_{N,\beta,\gamma,\Lambda}v-v|^2_{D(\mathcal{A}^k_{\beta,\gamma})}&=\left\|\sum_{l=N+1}^{\infty}\hat{v}_l^{\beta,\gamma}\mathcal{A}^k_{\beta,\gamma}\hat{H}_l^{\beta,\gamma}\right\|^2_{\omega_\gamma,\Lambda}\\&=\frac{\sqrt{\pi}}{\beta}\sum_{l=N+1}^{\infty}(\lambda_l^{\beta})^{2k}(\hat{v}_l^{\beta,\gamma})^2\\&=\frac{\beta}{\sqrt{\pi}}\sum_{l=N+1}^{\infty}(\lambda_l^{\beta})^{2k}(v,\hat{H}_l^{\beta,\gamma})^2_{\omega_\gamma,\Lambda}.\end{aligned}$$

Furthermore, we use (3.7) repeatedly to obtain

$$(v,\hat{H}_l^{\beta,\gamma})_{\omega_\gamma,\Lambda}=(\lambda_l^{\beta})^{-r}(\mathcal{A}^r_{\beta,\gamma}v,\hat{H}_l^{\beta,\gamma})_{\omega_\gamma,\Lambda}.$$

The above two equalities lead to that

$$\begin{aligned}|\hat{P}_{N,\beta,\gamma,\Lambda}v-v|^2_{D(A^k_{\beta,\gamma})}&=\frac{\beta}{\sqrt{\pi}}\sum_{l=N+1}^{\infty}(\lambda_l^{\beta})^{2k-2r}(\mathcal{A}^r_{\beta,\gamma}v,\hat{H}_l^{\beta,\gamma})^2_{\omega_\gamma,\Lambda}\\&\leq c\frac{\beta}{\sqrt{\pi}}(\lambda_{N+1}^{\beta})^{2k-2r}\sum_{l=N+1}^{\infty}(\mathcal{A}^r_{\beta,\gamma}v,\hat{H}_l^{\beta,\gamma})^2_{\omega_\gamma,\Lambda}\\&\leq c\frac{\beta}{\sqrt{\pi}}(\beta^2N)^{2k-2r}\sum_{l=0}^{\infty}(\mathcal{A}^r_{\beta,\gamma}v,\hat{H}_l^{\beta,\gamma})^2_{\omega_\gamma,\Lambda}.\end{aligned}$$

On the other hand, thanks to (3.2)–(3.4), we have

$$\|\mathcal{A}^r_{\beta,\gamma}v\|^2_{\omega_\gamma,\Lambda}=\frac{\beta}{\sqrt{\pi}}\sum_{l=0}^{\infty}(\mathcal{A}^r_{\beta,\gamma}v,\hat{H}_l^{\beta,\gamma})^2_{\omega_\gamma,\Lambda}.$$

Consequently,

$$|\hat{P}_{N,\beta,\gamma,\Lambda}v-v|_{D(\mathcal{A}^k_{\beta,\gamma})}\leq c(\beta^2N)^{k-r}\|\mathcal{A}^r_{\beta,\gamma}v\|_{\omega_\gamma,\Lambda}=c(\beta^2N)^{k-r}|v|_{D(\mathcal{A}^r_{\beta,\gamma})}.$$

The proof is completed. □

We next derive another error estimate of the orthogonal approximation $\hat{P}_{N,\beta,\gamma,\Lambda}v$.

Theorem 3.2. *If* $v \in H^r_{\omega_\gamma}(\Lambda)$ *and integers* $0 \le k \le r$, *then*

$$\|\hat{P}_{N,\beta,\gamma,\Lambda}v - v\|_{k,\omega_\gamma,\Lambda} \le c(\beta^2 N)^{\frac{k-r}{2}} \left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}. \tag{3.9}$$

Proof. Since $v \in H^r_{\omega_\gamma}(\Lambda) \subset L^2_{\omega_\gamma}(\Lambda)$, we have

$$\hat{P}_{N,\beta,\gamma,\Lambda}v(x) = \sum_{l=0}^{N} \hat{v}_l^{\beta,\gamma}\hat{H}_l^{\beta,\gamma}(x) = (1+x^2)^{\frac{\gamma}{2}} \sum_{l=0}^{N} \hat{v}_l^{\beta,\gamma} H_l^{\beta,\gamma}(x).$$

Let $P_{N,\beta,\Lambda}v$ be the same as in (2.4). With the aid of (3.4), it is easy to show that all coefficients $\hat{v}_l^{\beta,\gamma}$ are exactly the same as the coefficients of expansion (2.3) for the function $(1+x^2)^{-\frac{\gamma}{2}}v(x)$. In other words,

$$\hat{P}_{N,\beta,\gamma,\Lambda}v(x) = (1+x^2)^{\frac{\gamma}{2}} P_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v(x)\right).$$

Therefore, we use (2.5) to derive that

$$\begin{aligned}
\|\partial_x^k(\hat{P}_{N,\beta,\gamma,\Lambda}v - v)\|_{\omega_\gamma,\Lambda} &= \left\|\partial_x^k\left((1+x^2)^{\frac{\gamma}{2}}\left(P_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v\right) - (1+x^2)^{-\frac{\gamma}{2}}v\right)\right)\right\|_{\omega_\gamma,\Lambda} \\
&= \left\|\sum_{j=0}^{k} C_k^j \partial_x^{k-j}\left((1+x^2)^{\frac{\gamma}{2}}\right)\partial_x^j\left(P_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v\right) - (1+x^2)^{-\frac{\gamma}{2}}v\right)\right\|_{\omega_\gamma,\Lambda} \\
&\le c\sum_{j=0}^{k} C_k^j \left\|\partial_x^j\left(P_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v\right) - (1+x^2)^{-\frac{\gamma}{2}}v\right)\right\|_{\Lambda} \\
&\le c(\beta^2 N)^{\frac{k-r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}.
\end{aligned} \tag{3.10}$$

This ends the proof. □

In numerical analysis of the related spectral methods, we need the orthogonal approximation in the space $H^1_{\omega_\gamma}(\Lambda)$. The projection $\hat{P}^1_{N,\beta,\gamma,\Lambda} : H^1_{\omega_\gamma}(\Lambda) \to \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda)$ is defined by

$$(\partial_x(v - \hat{P}^1_{N,\beta,\gamma,\Lambda}v), \partial_x\varphi)_{\omega_\gamma,\Lambda} + (v - \hat{P}^1_{N,\beta,\gamma,\Lambda}v, \varphi)_{\omega_\gamma,\Lambda} = 0, \quad \forall \varphi \in \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda). \tag{3.11}$$

Theorem 3.3. *If* $v \in H^r_{\omega_\gamma}(\Lambda)$ *and integer* $r \ge 1$, *then*

$$\|v - \hat{P}^1_{N,\beta,\gamma,\Lambda}v\|_{1,\omega_\gamma,\Lambda} \le c(\beta^2 N)^{\frac{1-r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}. \tag{3.12}$$

Proof. By the projection theorem, we have

$$\|v - \hat{P}^1_{N,\beta,\gamma,\Lambda}v\|_{1,\omega_\gamma,\Lambda} \le \|v - \varphi\|_{1,\omega_\gamma,\Lambda}, \quad \forall \varphi \in \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda). \tag{3.13}$$

Let $w(x) = (1+x^2)^{-\frac{\gamma}{2}}v(x)$. If $v \in H^r_{\omega_\gamma}(\Lambda)$, then $w \in H^r(\Lambda)$. By taking $\varphi = (1+x^2)^{\frac{\gamma}{2}}P_{N,\beta,\Lambda}w \in \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda)$ in (3.13), we use (2.5) with $k = 0$ to derive that

$$\begin{aligned}
\|v - \varphi\|_{\omega_\gamma,\Lambda} &= \left\|(1+x^2)^{\frac{\gamma}{2}}w - (1+x^2)^{\frac{\gamma}{2}}P_{N,\beta,\Lambda}w\right\|_{\omega_\gamma,\Lambda} \\
&= \|w - P_{N,\beta,\Lambda}w\|_{\Lambda} \le c(\beta^2 N)^{-\frac{r}{2}}\|w\|_{H^r_{A,\beta}(\Lambda)} \\
&= c(\beta^2 N)^{-\frac{r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}.
\end{aligned} \tag{3.14}$$

Furthermore, we use (2.5) with $k = 0, 1$ to verify that

$$\begin{aligned}\|\partial_x(v-\varphi)\|_{\omega_\gamma,\Lambda} &= \left\|\partial_x\left((1+x^2)^{\frac{\gamma}{2}}w-(1+x^2)^{\frac{\gamma}{2}}P_{N,\beta,\Lambda}w\right)\right\|_{\omega_\gamma,\Lambda}\\ &\le \left\|(1+x^2)^{\frac{\gamma}{2}}\partial_x(w-P_{N,\beta,\Lambda}w)\right\|_{\omega_\gamma,\Lambda}+|\gamma|\left\|x(1+x^2)^{\frac{\gamma}{2}-1}(w-P_{N,\beta,\Lambda}w)\right\|_{\omega_\gamma,\Lambda}\\ &\le \|\partial_x(w-P_{N,\beta,\Lambda}w)\|_\Lambda+c\|w-P_{N,\beta,\Lambda}w\|_\Lambda\\ &\le c(\beta^2N)^{\frac{1-r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}. \end{aligned} \tag{3.15}$$

Finally, a combination of (3.13)–(3.15) leads to the desired result (3.12). □

3.3. Generalized Hermite–Gauss interpolation

We now turn to the new generalized Hermite–Gauss interpolation. Let $\sigma^\beta_{N,j}$ and $\omega^\beta_{N,j}$ be the same as in Section 2. We set

$$\hat\sigma^\beta_{N,j}=\sigma^\beta_{N,j},\qquad \hat\omega^{\beta,\gamma}_{N,j}=(1+(\hat\sigma^\beta_{N,j})^2)^{-\gamma}\omega^\beta_{N,j},\quad 0\le j\le N. \tag{3.16}$$

The corresponding discrete inner product and norm are as follows,

$$(u,v)_{N,\beta,\gamma,\Lambda}=\sum_{j=0}^N u(\hat\sigma^\beta_{N,j})v(\hat\sigma^\beta_{N,j})\hat\omega^{\beta,\gamma}_{N,j},\qquad \|v\|_{N,\beta,\gamma,\Lambda}=(v,v)^{\frac12}_{N,\beta,\gamma,\Lambda}.$$

For any $\phi\in\hat{\mathcal{Q}}^{\beta,\gamma}_m(\Lambda)$ and $\psi\in\hat{\mathcal{Q}}^{\beta,\gamma}_{2N+1-m}(\Lambda)$, there exist $q_\phi\in\mathcal{Q}^\beta_m(\Lambda)$ and $q_\psi\in\mathcal{Q}^\beta_{2N+1-m}(\Lambda)$, such that $\phi(x)=(1+x^2)^{\frac{\gamma}{2}}q_\phi(x)$ and $\psi(x)=(1+x^2)^{\frac{\gamma}{2}}q_\psi(x)$, respectively. Thereby, we use (2.6) to deduce that for any $\phi\in\hat{\mathcal{Q}}^{\beta,\gamma}_m(\Lambda)$ and $\psi\in\hat{\mathcal{Q}}^{\beta,\gamma}_{2N+1-m}(\Lambda)$,

$$\begin{aligned}(\phi,\psi)_{\omega_\gamma,\Lambda}&=(q_\phi,q_\psi)_\Lambda\\ &=\sum_{j=0}^N q_\phi(\sigma^\beta_{N,j})q_\psi(\sigma^\beta_{N,j})\omega^\beta_{N,j}\\ &=\sum_{j=0}^N \phi(\hat\sigma^\beta_{N,j})\psi(\hat\sigma^\beta_{N,j})\hat\omega^{\beta,\gamma}_{N,j}=(\phi,\psi)_{N,\beta,\gamma,\Lambda}.\end{aligned} \tag{3.17}$$

In particular,

$$\|\phi\|_{\omega_\gamma,\Lambda}=\|\phi\|_{N,\beta,\gamma,\Lambda},\quad \forall\,\phi\in\hat{\mathcal{Q}}^{\beta,\gamma}_N(\Lambda). \tag{3.18}$$

For any $v\in C(\Lambda)$, the new generalized Hermite–Gauss interpolation $\hat I_{N,\beta,\gamma,\Lambda}v\in\hat{\mathcal{Q}}^{\beta,\gamma}_N(\Lambda)$ is determined uniquely by

$$\hat I_{N,\beta,\gamma,\Lambda}v(\hat\sigma^\beta_{N,j})=v(\hat\sigma^\beta_{N,j}),\quad 0\le j\le N. \tag{3.19}$$

Theorem 3.4. *If $v\in H^r_{\omega_\gamma}(\Lambda)$, $(1+x^2)^{-\frac{\gamma}{2}}v\in H^r_{A,\beta}(\Lambda)$, integers $r\ge 1$ and $0\le k\le r$, then*

$$\|\hat I_{N,\beta,\gamma,\Lambda}v-v\|_{k,\omega_\gamma,\Lambda}\le c(\beta^k+1)(\beta^2N)^{\frac13+\frac{k-r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}. \tag{3.20}$$

Proof. We have from (2.7) and (3.19) that

$$(1+(\hat\sigma^\beta_{N,j})^2)^{-\frac{\gamma}{2}}\hat I_{N,\beta,\gamma,\Lambda}v(\hat\sigma^\beta_{N,j})=I_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v(x)\right)\Big|_{x=\hat\sigma^\beta_{N,j}},\quad 0\le j\le N.$$

Moreover, both of $(1+x^2)^{-\frac{\gamma}{2}}\hat I_{N,\beta,\gamma,\Lambda}v(x)$ and $I_{N,\beta,\Lambda}((1+x^2)^{-\frac{\gamma}{2}}v(x))$ belong to the same finite-dimensional set $\mathcal{Q}^\beta_N(\Lambda)$. The above facts imply

$$\hat I_{N,\beta,\gamma,\Lambda}v(x)=(1+x^2)^{\frac{\gamma}{2}}I_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v(x)\right).$$

Table 1
The global weighted errors with $\beta = 1, \gamma = \alpha + \frac{7}{12}$.

	$\alpha = -3$	$\alpha = -1$	$\alpha = 0$	$\alpha = 1$	$\alpha = 3$
$k = 1$	2.63E−15	4.47E−15	7.74E−15	1.57E−14	8.35E−14
$k = 2$	3.21E−15	5.48E−15	9.78E−15	1.97E−14	9.89E−14
$k = 3$	4.12E−15	6.21E−15	9.57E−15	1.76E−14	7.85E−14

Table 2
The point-wise errors with $\beta = 1, \gamma = \alpha + \frac{7}{12}$.

	$\alpha = -3$	$\alpha = -1$	$\alpha = 0$	$\alpha = 1$	$\alpha = 3$
$k = 1$	1.33E−15	3.39E−15	9.54E−15	5.50E−14	2.05E−12
$k = 2$	2.88E−15	4.44E−15	1.29E−14	8.08E−14	2.96E−12
$k = 3$	3.77E−15	4.27E−15	1.15E−14	6.22E−14	2.19E−12

Consequently, we use (2.10) to verify that

$$\begin{aligned}\|\partial_x^k(\hat{I}_{N,\beta,\gamma,\Lambda}v - v)\|_{\omega_\gamma,\Lambda} &= \left\|\partial_x^k\left((1+x^2)^{\frac{\gamma}{2}}\left(I_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v\right) - (1+x^2)^{-\frac{\gamma}{2}}v\right)\right)\right\|_{\omega_\gamma,\Lambda} \\ &= \left\|\sum_{j=0}^{k} C_k^j \partial_x^{k-j}\left((1+x^2)^{\frac{\gamma}{2}}\right)\partial_x^j\left(I_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v\right) - (1+x^2)^{-\frac{\gamma}{2}}v\right)\right\|_{\omega_\gamma,\Lambda} \\ &\le c\sum_{j=0}^{k} C_k^j \left\|\partial_x^j\left(I_{N,\beta,\Lambda}\left((1+x^2)^{-\frac{\gamma}{2}}v\right) - (1+x^2)^{-\frac{\gamma}{2}}v\right)\right\|_{\Lambda} \\ &\le c(\beta^k+1)(\beta^2 N)^{\frac{1}{3}+\frac{k-r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}.\end{aligned}$$

This completes the proof. □

Finally, by using (3.19), (3.17) and (3.20) successively, we derive that

$$\begin{aligned}|(v,\phi)_{\omega_\gamma,\Lambda} - (v,\phi)_{N,\beta,\gamma,\Lambda}| &= |(v - \hat{I}_{N,\beta,\gamma,\Lambda}v, \phi)_{\omega_\gamma,\Lambda}| \\ &\le c\|v - \hat{I}_{N,\beta,\gamma,\Lambda}v\|_{\omega_\gamma,\Lambda}\|\phi\|_{\omega_\gamma,\Lambda} \\ &\le c(\beta^2 N)^{\frac{1}{3}-\frac{r}{2}}\left\|(1+x^2)^{-\frac{\gamma}{2}}v\right\|_{H^r_{A,\beta}(\Lambda)}\|\phi\|_{\omega_\gamma,\Lambda}, \quad \forall\phi\in\hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda). \end{aligned} \tag{3.21}$$

3.4. Numerical test

We now check the efficiency of the new approximation given by (3.5). We consider the test function

$$v(x) = (x^2+x+1)^{\frac{\alpha}{2}}\sin kx, \tag{3.22}$$

which oscillates as $|x|$ increases. Moreover, its amplitude decays to zero for $\alpha < 0$, and grows to the infinity for $\alpha > 0$, as $|x|$ increases. This test function belongs to the weighted space $L^2_{|x|^\mu}(\Lambda)$ as long as $\mu < -2\alpha - 1$. Since the new generalized Hermite functions $H_l^{\beta,\gamma}(x)$ are mutually orthogonal with the weight function $\omega_\gamma(x) = (1+x^2)^{-\gamma}$, the above test function could be approximated by the generalized Hermite orthogonal approximation defined by (3.5), with $\gamma \ge -\frac{1}{2}\mu > \alpha + \frac{1}{2}$.

In Table 1, we show the global $\omega_\gamma(x)$-weighted errors $\|v - \hat{P}_{N,\beta,\gamma,\Lambda}v\|_{\omega_\gamma,\Lambda}$ with $\beta = 1, \gamma = \alpha + \frac{7}{12}$ and the mode $N = 20$. We find that the new approximation with moderate mode N fits the approximated function well. In Table 2, we list the point-wise errors $\max_{0\le j\le N}|v(\hat{\sigma}^\beta_{N,j}) - \hat{P}_{N,\beta,\gamma,\Lambda}v(\hat{\sigma}^\beta_{N,j})|$. Clearly, the new generalized Hermite approximation also possesses small point-wise numerical errors.

In Tables 3 and 4, we list the global weighted errors and the point-wise errors with $\beta = 1, 1.5$, respectively. We see from Tables 1–4 that the suitable choice of parameter β leads to better numerical results sometimes.

Remark 3.1. If we use the standard Hermite orthogonal approximation for the test function (3.22), then there exists the weight function e^{-x^2}, which is much stronger than the weight function $\omega_\gamma(x) = (1+x^2)^{-\gamma}$ used in our new approximation. Therefore, although its global weighted errors are smaller than those of the approximation (3.5), its point-wise errors might be bigger than those of the approximation (3.5). In Tables 5 and 6, we list the global e^{-x^2}-weighted errors and the point-wise errors of the standard Hermite orthogonal approximation with the mode $N = 20$, respectively. We find from Tables 1, 2, 5

Table 3
The global weighted errors with $\gamma = \alpha + \frac{7}{12}$.

	$\alpha = 1$			$\alpha = 3$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$\beta = 1$	1.57E−14	1.97E−14	1.76E−14	8.35E−14	9.89E−14	7.85E−14
$\beta = 1.5$	9.66E−15	1.16E−14	1.05E−14	1.19E−14	1.36E−14	1.15E−14

Table 4
The point-wise errors with $\gamma = \alpha + \frac{7}{12}$.

	$\alpha = 1$			$\alpha = 3$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$\beta = 1$	5.50E−14	8.08E−14	6.22E−14	2.05E−12	2.96E−12	2.19E−12
$\beta = 1.5$	3.73E−14	6.04E−14	5.77E−14	6.96E−13	1.09E−12	1.09E−12

Table 5
The global e^{-x^2}-weighted errors of standard Hermite approximation.

	$\alpha = -3$	$\alpha = -1$	$\alpha = 0$	$\alpha = 1$	$\alpha = 3$
$k = 1$	8.70E−16	1.23E−15	1.61E−15	2.54E−15	1.11E−14
$k = 2$	1.39E−15	2.02E−15	2.57E−15	2.98E−15	9.40E−15
$k = 3$	1.91E−15	2.09E−15	2.64E−15	3.77E−15	1.06E−14

Table 6
The point-wise errors of standard Hermite approximation.

	$\alpha = -3$	$\alpha = -1$	$\alpha = 0$	$\alpha = 1$	$\alpha = 3$
$k = 1$	9.41E−11	2.43E−10	1.70E−10	2.54E−10	3.76E−10
$k = 2$	2.54E−10	2.11E−10	2.20E−10	3.00E−10	1.83E−9
$k = 3$	4.63E−10	1.43E−10	2.04E−10	3.86E−10	2.15E−9

and 6 that the standard Hermite orthogonal approximation has smaller global weighted errors, but larger point-wise errors, than the new approximation (3.5).

Remark 3.2. As pointed out by the referee of this paper, for the test function $v(x)$ given by (3.22), we may follow the idea of London to approximate the auxiliary function $v^*(x) = v(x)\,\mathrm{sech}\,x$ by using the Hermite functions. Let $\hat{v}_l^*$ be the coefficients of the expansion of $v^*(x)$ in terms of $e^{-\frac{1}{2}x^2}H_l(x)$. Then we obtain the following approximation to the original function,

$$P_N^* v(x) = \frac{1}{\mathrm{sech}\,x}\left(\sum_{l=0}^{N} \hat{v}_l^* e^{-\frac{1}{2}x^2} H_l(x)\right). \tag{3.23}$$

We could use (2.5) with $k = 0$ and $\beta = 1$ to verify that

$$\|P_N^* v - v\|_{\mathrm{sech}^2 x,\Lambda} = \|P_{N,1,\Lambda} v^* - v^*\|_\Lambda \le cN^{-\frac{r}{2}} \|v\,\mathrm{sech}\,x\|_{H^r_{A,1}(\Lambda)}.$$

Thus, the global errors with the weight function $\mathrm{sech}^2 x$ are small usually. But the corresponding global errors with the weight function $\omega_\gamma(x)$ are bigger than those of the approximation (3.5). In Tables 7 and 8, we list the global $\omega_\gamma(x)$-weighted errors and the point-wise errors of the approximation (3.23) with the mode $N = 20$, respectively. They confirm the analysis. We also find that the point-wise errors of the approximations (2.3) and (3.23) are nearly the same for the test function (3.22) with $\alpha > 0$, while the point-wise errors of the approximation (3.5) are smaller for the test function (3.22) with $\alpha < 0$. By the way, in applications of the above two approximations to numerical solutions of differential equations defined on the whole line, we have to multiply the underlying differential equations by the weight functions and integrate the resulting equalities by parts, and then derive their weak formulations. Moreover, in the numerical analysis of the corresponding spectral methods, we need some results on the $H^1_{\mathrm{sech}^2 x}(\Lambda)$-orthogonal approximation and the $H^1_{\omega_\gamma}(\Lambda)$-orthogonal approximation, respectively. For this purpose, it seems simpler to use the approximation with the weight function $\omega_\gamma(x)$ usually.

Remark 3.3. We may also use the generalized Jacobi rational approximation proposed in [12]. For $a, b > -1$, $J_l^{(a,b)}(x)$ stands for the Jacobi polynomial of degree l. For any real numbers a and b,

$$\hat{a} := \begin{cases} -a, & a \le -1, \\ 0, & a > -1, \end{cases} \qquad \bar{a} := \begin{cases} -a, & a \le -1, \\ a, & a > -1, \end{cases}$$

Table 7
The global $\omega_\gamma(x)$-weighted errors of approximation (3.23).

	$\alpha=-3$	$\alpha=-1$	$\alpha=0$	$\alpha=1$	$\alpha=3$
$k=1$	3.91E−15	5.64E−15	1.51E−14	7.93E−14	2.49E−12
$k=2$	6.95E−15	9.10E−15	3.06E−14	1.15E−13	3.21E−12
$k=3$	1.91E−14	1.47E−14	3.23E−14	1.09E−13	2.51E−12

Table 8
The point-wise errors of approximation (3.23).

	$\alpha=-3$	$\alpha=-1$	$\alpha=0$	$\alpha=1$	$\alpha=3$
$k=1$	2.43E−15	3.86E−15	9.65E−15	5.53E−14	3.11E−12
$k=2$	3.70E−15	5.66E−15	2.38E−14	7.23E−14	2.99E−12
$k=3$	1.09E−14	1.11E−14	1.76E−14	7.94E−14	2.31E−12

Table 9
The global $\omega_R^{(\alpha-\frac{11}{12},\alpha-\frac{11}{12})}(x)$-weighted errors of approximation (3.24).

	$\alpha=-3$	$\alpha=-1$	$\alpha=0$	$\alpha=1$	$\alpha=3$
$k=1$	7.01E−15	4.36E−15	7.68E−15	1.56E−14	7.15E−14
$k=2$	5.01E−15	5.41E−15	9.36E−15	2.01E−14	9.81E−14
$k=3$	3.08E−15	6.17E−15	8.16E−15	1.68E−14	7.76E−14

Table 10
The point-wise errors of approximation (3.24).

	$\alpha=-3$	$\alpha=-1$	$\alpha=0$	$\alpha=1$	$\alpha=3$
$k=1$	1.34E−15	3.56E−15	9.16E−15	5.51E−14	2.01E−12
$k=2$	2.16E−15	4.40E−15	1.36E−14	8.70E−14	2.38E−12
$k=3$	3.71E−15	4.26E−15	1.18E−14	6.11E−14	2.25E−12

(likewise for $\hat{b}$ and $\bar{b}$). The symbol $[a]$ represents the largest integer $\le a$. The generalized Jacobi functions are given by

$$\tilde{J}_l^{(a,b)}(x) := \begin{cases} J_l^{(a,b)}(x), & a,b>-1,\\ (1+x)^{-b}J_{l-[-b]}^{(a,-b)}(x), & a>-1,\ b\le -1,\\ (1-x)^{-a}J_{l-[-a]}^{(-a,b)}(x), & a\le -1,\ b>-1,\\ (1-x)^{-a}(1+x)^{-b}J_{l-[-a]-[-b]}^{(-a,-b)}(x), & a,b\le -1.\end{cases}$$

The generalized Jacobi rational functions are defined by

$$R_l^{(a,b)}(x) = \tilde{J}_l^{(a,b)}\left(\frac{x}{\sqrt{x^2+1}}\right), \quad l\ge [\hat{a}]+[\hat{b}].$$

The weight function is

$$\omega_R^{(a,b)}(x) = (\sqrt{x^2+1}+x)^{b-a}(x^2+1)^{-\frac{a+b+3}{2}}.$$

The generalized Jacobi rational functions form a complete $L^2_{\omega_R^{(a,b)}}(\Lambda)$-orthogonal system.

We now approximate the test function (3.22) in a specific way, namely,

$$P_N^{**}v(x) = \sum_{l=0}^{N} \hat{v}_l^{**} R_l^{\left(\alpha-\frac{11}{12},\alpha-\frac{11}{12}\right)}(x), \tag{3.24}$$

v_l^{**} being the coefficients of the expansion of $v(x)$ in terms of $R_l^{(\alpha-\frac{11}{12},\alpha-\frac{11}{12})}(x)$. In Tables 9 and 10, we list the global $\omega_R^{(\alpha-\frac{11}{12},\alpha-\frac{11}{12})}(x)$-weight errors and the point-wise errors of the approximation (3.24) with the mode $N=20$, respectively. Clearly, the weight function $\omega_R^{(\alpha-\frac{11}{12},\alpha-\frac{11}{12})}(x)$ is exactly the same as the weight function $\omega_\gamma(x)$, $\gamma=\alpha+\frac{7}{12}$. By comparing Tables 1, 2, 9 and 10, we find that the approximations (3.5) and (3.24) have nearly the same accuracy. However, in their applications to numerical solutions of differential equations, we have to derive the weak formulations of underlying problems, and need some results on the $H^1_{\omega_R^{(\alpha-\frac{11}{12},\alpha-\frac{11}{12})}}(\Lambda)$-orthogonal approximation and the $H^1_{\omega_\gamma}(\Lambda)$-orthogonal approximation, respectively. It seems simpler to use the approximation (3.5) than the approximation (3.24). Besides, in actual computation, it is easier to perform the generalized Hermite orthogonal approximation than the generalized Jacobi rational orthogonal approximation, except the standard Chebyshev rational orthogonal approximation which is only available for the functions behaving like $|x|^\alpha$, $\alpha < 0$, for large $|x|$.

4. Generalized Hermite spectral method

In this section, we propose the generalized Hermite spectral method.

4.1. A linear problem on the whole line

Let λ be a positive constant, and $f \in L^2_{\omega_\gamma}(\Lambda)$, $\gamma > \alpha + \frac{1}{2}$. We consider the following model problem,

$$\begin{cases} -\partial_x^2 U(x) + \lambda U(x) = f(x), & x \in \Lambda, \\ U(x)|x|^{-\alpha} \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{4.1}$$

Let $V(\Lambda) = H^1_{\omega_\gamma}(\Lambda)$. We introduce the bilinear form

$$\begin{aligned} \mathcal{A}_{\lambda,\gamma,\Lambda}(u, v) = &\int_\Lambda \partial_x u(x)\partial_x v(x)(1+x^2)^{-\gamma}dx - 2\gamma \int_\Lambda \partial_x u(x)v(x)x(1+x^2)^{-\gamma-1}dx \\ &+ \lambda \int_\Lambda u(x)v(x)(1+x^2)^{-\gamma}dx, \quad \forall u,\ v \in V(\Lambda). \end{aligned} \tag{4.2}$$

With the aid of the Cauchy inequality, a direct calculation shows that

$$|\mathcal{A}_{\lambda,\gamma,\Lambda}(u, v)| \le \left(\frac{1}{2} + |\gamma|\right) \|\partial_x u\|^2_{\omega_\gamma,\Lambda} + \frac{\lambda}{2}\|u\|^2_{\omega_\gamma,\Lambda} + \frac{1}{2}\|\partial_x v\|^2_{\omega_\gamma,\Lambda} + \left(\frac{\lambda}{2} + |\gamma|\right)\|v\|^2_{\omega_\gamma,\Lambda}, \quad \forall u,\ v \in V(\Lambda). \tag{4.3}$$

For simplicity of statements, we set

$$c_\gamma = \begin{cases} -\gamma, & \text{for } -1 \le \gamma \le 0, \\ \gamma(2\gamma+1), & \text{otherwise.} \end{cases}$$

Then, for any $v \in H^2_{\omega_\gamma}(\Lambda)$,

$$\begin{aligned} -2\gamma \int_\Lambda \partial_x v(x)v(x)x(1+x^2)^{-\gamma-1}dx &= \gamma \int_\Lambda v^2(x)(1+x^2)^{-\gamma-1}dx - 2\gamma(\gamma+1)\int_\Lambda v^2(x)x^2(1+x^2)^{-\gamma-2}dx \\ &\ge -c_\gamma \|v\|^2_{\omega_\gamma,\Lambda}. \end{aligned} \tag{4.4}$$

Inserting (4.4) into (4.2) with $u = v$, we obtain

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(v, v) \ge \|\partial_x v\|^2_{\omega_\gamma,\Lambda} + (\lambda - c_\gamma)\|v\|^2_{\omega_\gamma,\Lambda}. \tag{4.5}$$

We now derive another property of the bilinear form $\mathcal{A}_{\lambda,\gamma,\lambda}(u, v)$, which plays an important role in the numerical analysis of the spectral method for the whole line. Let $W(\Lambda) \subseteqq V(\Lambda)$, and $\mathcal{Q}^*_N(\Lambda) \subset V(\Lambda)$ be a finite-dimensional subspace. Furthermore, $W_N(\Lambda) = W(\Lambda) \cap \mathcal{Q}^*_N(\Lambda)$. We define the operator ${}_*P^1_{N,\lambda,\beta,\gamma,\Lambda} : W(\Lambda) \to W_N(\Lambda)$, by

$$\mathcal{A}_{\lambda,\gamma,\Lambda}({}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - v, \phi) = 0, \quad \forall\, \phi \in W_N(\Lambda). \tag{4.6}$$

Proposition 4.1. *If $v \in W(\Lambda)$, $w \in W_N(\Lambda)$ and $\lambda > c_\gamma$, then*

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(v - {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v, v - *P^1_{N,\lambda,\beta,\gamma,\Lambda}v) \le \mathcal{A}_{\lambda,\gamma,\Lambda}(v - w, v - w). \tag{4.7}$$

Proof. A direct calculation shows

$$\begin{aligned} \mathcal{A}_{\lambda,\gamma,\Lambda}(v - w, v - w) = &\mathcal{A}_{\lambda,\gamma,\Lambda}(v - {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v, v - {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v) \\ &+ \mathcal{A}_{\lambda,\gamma,\Lambda}({}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - w, {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - w) \\ &+ 2\mathcal{A}_{\lambda,\gamma,\Lambda}(v - {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v, {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - w). \end{aligned}$$

Thanks to (4.6), we have

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(v - {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v, {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - w) = 0.$$

Due to $\lambda > c_\gamma$, we use (4.5) to assert that

$$\mathcal{A}_{\lambda,\gamma,\Lambda}({}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - w, {}_*P^1_{N,\lambda,\beta,\gamma,\Lambda}v - w) \ge 0.$$

Finally, the desired inequality (4.7) follows from the previous statements immediately. $\square$

Now, let $v \in V(\Lambda)$. By multiplying (4.1) by $v(x)\omega_\gamma(x)$ and integrating the resulting equation by parts, we derive a weak formulation of (4.1). It is to look for the solution $U \in V(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(U, v) = (f, v)_{\omega_\gamma,\Lambda}, \quad \forall v \in V(\Lambda). \tag{4.8}$$

If $\lambda > c_\gamma$, then we use (4.3), (4.5) and the Lax–Milgram lemma to verify that problem (4.8) admits a unique solution.

For solving the above problem numerically, we introduce the finite-dimensional space

$$V_N(\Lambda) = V(\Lambda) \cap \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda).$$

The spectral method for solving problem (4.8) is to seek the solution $u_N \in V_N(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(u_N, \phi) = (f, \phi)_{\omega_\gamma,\Lambda}, \quad \forall \phi \in V_N(\Lambda). \tag{4.9}$$

For checking the existence of solutions of (4.9), it suffices to prove the uniqueness of its solutions. Assume that $u_N^{(1)}(x)$ and $u_N^{(2)}(x)$ are solutions of (4.9), and $\tilde{u}_N(x) = u_N^{(1)}(x) - u_N^{(2)}(x) \in V_N(\Lambda)$. Then

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(\tilde{u}_N, \phi) = 0, \quad \forall\, \phi \in V_N(\Lambda).$$

Putting $\phi = \tilde{u}_N \in V_N(\Lambda)$ in the above equation, we use (4.5) to obtain

$$\|\partial_x \tilde{u}_N\|^2_{\omega_\gamma,\Lambda} + (\lambda - c_\gamma)\|\tilde{u}_N\|^2_{\omega_\gamma,\Lambda} \le \mathcal{A}_{\lambda,\gamma,\Lambda}(\tilde{u}_N, \tilde{u}_N) = 0.$$

If $\lambda > c_\gamma$, then $\tilde{u}_N(x) \equiv 0$. This means the uniqueness of the solution of (4.9).

We now estimate the error of the numerical solution $u_N(x)$. To do this, we introduce the auxiliary operator $\overline{P}^1_{N,\beta,\gamma,\Lambda} : V(\Lambda) \to V_N(\Lambda)$, defined by

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(\overline{P}^1_{N,\beta,\gamma,\Lambda} v - v, \phi) = 0, \quad \forall\, \phi \in V_N(\Lambda). \tag{4.10}$$

We have from (4.8) and (4.10) that

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(\overline{P}^1_{N,\beta,\gamma,\Lambda} U, \phi) = (f, \phi)_{\omega_\gamma,\Lambda}, \quad \forall \phi \in V_N(\Lambda). \tag{4.11}$$

Subtracting (4.11) from (4.9), yields

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(u_N - \overline{P}^1_{N,\beta,\gamma,\Lambda} U, \phi) = 0, \quad \forall \phi \in V_N(\Lambda).$$

Taking $\phi = u_N - \overline{P}^1_{N,\beta,\gamma,\Lambda} U$ in the above equation, we obtain

$$\mathcal{A}_{\lambda,\gamma,\Lambda}(u_N - \overline{P}^1_{N,\beta,\gamma,\Lambda} U, u_N - \overline{P}^1_{N,\beta,\gamma,\Lambda} U) = 0.$$

This fact, together with (4.5), implies $u_N = \overline{P}^1_{N,\beta,\gamma,\Lambda} U$.

So far, it remains to estimate the approximation error of the auxiliary operator $\overline{P}^1_{N,\beta,\gamma,\Lambda} U$. For this purpose, we use Proposition 4.1 with

$$W(\Lambda) = V(\Lambda), \qquad W_N(\Lambda) = V_N(\Lambda), \quad v = U,$$
$$w = \hat{P}^1_{N,\beta,\gamma,\Lambda} U, \qquad {}_*P^1_{N,\beta,\gamma,\Lambda} U = \overline{P}^1_{N,\beta,\gamma,\Lambda} U.$$

Then, by virtue of (4.5), (4.7) and (4.3), we verify that

$$\begin{aligned}
&\|\partial_x(U - \overline{P}^1_{N,\beta,\gamma,\Lambda} U)\|^2_{\omega_\gamma,\Lambda} + (\lambda - c_\gamma)\|U - \overline{P}^1_{N,\beta,\gamma,\Lambda} U\|^2_{\omega_\gamma,\Lambda} \\
&\quad \le \mathcal{A}_{\lambda,\gamma,\Lambda}(U - \overline{P}^1_{N,\beta,\gamma,\Lambda} U, U - \overline{P}^1_{N,\beta,\gamma,\Lambda} U) \\
&\quad \le \mathcal{A}_{\lambda,\gamma,\Lambda}(U - \hat{P}^1_{N,\beta,\gamma,\Lambda} U, U - \hat{P}^1_{N,\beta,\gamma,\Lambda} U) \\
&\quad \le (1 + |\gamma|)\|\partial_x(U - \hat{P}^1_{N,\beta,\gamma,\Lambda} U)\|^2_{\omega_\gamma,\Lambda} + (\lambda + |\gamma|)\|U - \hat{P}^1_{N,\beta,\gamma,\Lambda} U\|^2_{\omega_\gamma,\Lambda}.
\end{aligned} \tag{4.12}$$

Finally, by using (4.12) and (3.12) successively, we deduce that if $\lambda > c_\gamma$ and integer $r \ge 2$, then

$$\begin{aligned}
\|U - u_N\|^2_{H^1_{\omega_\gamma}(\Lambda)} &= \|\partial_x(U - \overline{P}^1_{N,\beta,\gamma,\Lambda} U)\|^2_{\omega_\gamma,\Lambda} + \|U - \overline{P}^1_{N,\beta,\gamma,\Lambda} U\|^2_{\omega_\gamma,\Lambda} \\
&\le \left(1 + \frac{1}{\lambda - c_\gamma}\right)((1 + |\gamma|)\|\partial_x(U - \hat{P}^1_{N,\beta,\gamma,\Lambda} U)\|^2_{\omega_\gamma,\Lambda} + (\lambda + |\gamma|)\|U - \hat{P}^1_{N,\beta,\gamma,\Lambda} U\|^2_{\omega_\gamma,\Lambda}) \\
&\le c(\beta^2 N)^{\frac{1-r}{2}} \left\|(1 + x^2)^{-\frac{\gamma}{2}} U\right\|_{H^r_{A,\beta}(\Lambda)}.
\end{aligned} \tag{4.13}$$

Table 11
The point-wise error of algorithm (4.14) with $\beta = 1$.

	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$	$N = 120$	$N = 140$
$\gamma = 0$	6.05E−2	4.15E−3	4.82E−4	7.90E−5	1.93E−5	5.25E−6	1.56E−6
$\gamma = -2$	5.46E−2	3.51E−3	3.90E−4	6.16E−5	1.39E−5	3.74E−6	1.10E−6
$\gamma = -3$	4.69E−2	2.81E−3	2.97E−4	4.39E−5	9.44E−6	2.19E−6	7.62E−7

Table 12
The point-wise error of algorithm (4.14) with $\gamma = -3$.

	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$	$N = 120$	$N = 140$
$\beta = 1$	4.69E−2	2.81E−3	2.97E−4	4.39E−5	9.44E−6	2.19E−6	7.62E−7
$\beta = 1.5$	1.56E−3	3.29E−5	4.86E−6	1.76E−6	1.30E−6	5.40E−7	4.13E−7

We next describe the implementation for the spectral scheme (4.9). Let

$$\phi_l(x) = \pi^{-\frac{1}{4}} \hat{H}_l^{\beta,\gamma}(x), \quad 0 \le l \le N.$$

We expand the numerical solution as

$$u_N(x) = \sum_{l=0}^{N} \hat{u}_l \phi_l(x).$$

Inserting the above expression into (4.9) with $\phi = \phi_k(x)$, we obtain

$$\sum_{l=0}^{N} \Big((\partial_x \phi_l, \partial_x \phi_k)_{\omega_\gamma,\Lambda} - 2\gamma (\partial_x \phi_l, x(1+x^2)^{-1}\phi_k)_{\omega_\gamma,\Lambda} + \lambda(\phi_l, \phi_k)_{\omega_\gamma,\Lambda} \Big) \hat{u}_l = (f, \phi_k)_{\omega_\gamma,\Lambda}, \quad 0 \le k \le N. \tag{4.14}$$

We can rewrite system (4.14) as a compact matrix form. To do this, we introduce the matrices $A = (a_{k,l})_{0\le k,l\le N}$, $B = (b_{k,l})_{0\le k,l\le N}$ and $C = (c_{k,l})_{0\le k,l\le N}$, with the following entries:

$$a_{k,l} = (\partial_x \phi_l, \partial_x \phi_k)_{\omega_\gamma,\Lambda}, \qquad b_{k,l} = (\partial_x \phi_l, x(1+x^2)^{-1}\phi_k)_{\omega_\gamma,\Lambda}, \qquad c_{k,l} = (\phi_l, \phi_k)_{\omega_\gamma,\Lambda}.$$

Furthermore, let $\hat{\mathbf{u}} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_N)^T$ and $\mathbf{F} = (F_0, F_1, \ldots, F_N)^T$ with $F_k = (f, \phi_k)_{\omega_\gamma,\Lambda}$. Then, system (4.14) becomes

$$(A - 2\gamma B + C)\hat{\mathbf{u}} = \mathbf{F}. \tag{4.15}$$

We now use the algorithm (4.14) to solve problem (4.9) with the test function

$$U(x) = (x^2+1)^{\frac{\alpha}{2}} \sin kx.$$

Clearly, $U \in L^2_{\omega_\gamma}(\Lambda)$ as long as $\gamma > \alpha + \frac{1}{2}$.

In actual computation, we take $\lambda = 1$. We measure the numerical accuracy by the point-wise error $E_{N,pw} = \max_{0\le j\le N} |U(\hat{\sigma}^{\beta}_{N,j}) - u_N(\hat{\sigma}^{\beta}_{N,j})|$.

In Table 11, we list the values of $E_{N,pw}$, with $k = 2$, $\alpha = -6$, $\beta = 1$ and $\gamma = 0, -2, -3$, vs. the mode N. The numerical results demonstrate the convergence of algorithm (4.14), as predicted by (4.13). They also show that the numerical results with $\gamma = -3$ are better than the results with $\gamma = 0$. In fact, the case with $\gamma = 0$ corresponds to the spectral method using the Hermite functions given in [6]. Because the base functions with $\gamma = -3$ simulate the asymptotic behavior of the test function more reasonably than the base functions with $\gamma = 0$, so the numerical results with $\gamma = -3$ are better than those with $\gamma = 0$. In Table 12, we list the corresponding values of $E_{N,pw}$ with $\beta = 1, 1.5$ and $\gamma = -3$. They indicate that a suitable choice of parameter β provides better numerical results sometimes.

4.2. Sine–Gordon equation

As an example of nonlinear problems, we consider the following sine–Gordon equation,

$$\begin{cases} \partial_t^2 U(x,t) - \partial_x^2 U(x,t) + \sin U(x,t) = f(x,t), & x \in \Lambda,\ 0 < t \le T, \\ \partial_t U(x,0) = U_1(x), \qquad U(x,0) = U_0(x), & x \in \Lambda. \end{cases} \tag{4.16}$$

In addition, $U(x)$ satisfies certain boundary conditions at infinity.

We now derive a weak formulation of (4.16), which depends on the asymptotic behavior of $U(x,t)$ at infinity. We suppose that

$$U(x,t)\partial_x U(x,t)\omega_\gamma(x) \to 0, \quad \text{as } |x| \to \infty,\ 0 \le t \le T,\ \text{a.e.} \tag{4.17}$$

Indeed, if $U \in H^1_{\omega_\gamma}(\Lambda)$, $0 \le t \le T$, a.e., then $U(x,t)\omega_\gamma^{\frac{1}{2}}(x) = o(x^{-\frac{1}{2}})$ and $\partial_x U(x,t)\omega_\gamma^{\frac{1}{2}}(x) = o(x^{-\frac{1}{2}})$ as $|x| \to \infty$, $0 \le t \le T$, a.e. In this case, the boundary condition (4.17) is fulfilled.

Let $V(\Lambda) = H^1_{\omega_\gamma}(\Lambda)$ as in the last subsection. We multiply the first equation of (4.16) by $v(x)\omega_\gamma(x)v$, $v \in V(\Lambda)$, and integrate the resulting equality by parts over the interval Λ. Then we obtain the weak formulation of problem (4.16) with the boundary condition (4.17). It is to find the solution $U \in W^{1,\infty}(0,T; L^2_{\omega_\gamma}(\Lambda)) \cap L^\infty(0,T; V(\Lambda))$ such that

$$\begin{cases} (\partial_t^2 U(t), v)_{\omega_\gamma,\Lambda} + (\partial_x U(t), \partial_x(v\omega_\gamma))_\Lambda + (\sin U(t), v)_{\omega_\gamma,\Lambda} = (f(t), v)_{\omega_\gamma,\Lambda}, \quad \forall v \in V(\Lambda),\ 0 < t \le T, \\ \partial_t U(x,0) = U_1(x), \qquad U(x,0) = U_0(x), \quad x \in \Lambda. \end{cases} \tag{4.18}$$

If $U_0 \in V(\Lambda)$, $U_1 \in L^2_{\omega_\gamma}(\Lambda)$ and $f \in L^2(0,T; L^2_{\omega_\gamma}(\Lambda))$, then problem (4.18) admits a unique solution.

Let $V_N(\Lambda) = V(\Lambda) \cap \hat{\mathcal{Q}}_N^{\beta,\gamma}(\Lambda)$ as before. The spectral scheme for solving problem (4.18) is to seek $u_N(t) \in V_N(\Lambda)$ for all $t \ge 0$, such that

$$\begin{cases} (\partial_t^2 u_N(t), \phi)_{\omega_\gamma,\Lambda} + (\partial_x u_N(t), \partial_x(\omega_\gamma\phi))_\Lambda + (\sin u_N(t), \phi)_{\omega_\gamma,\Lambda} = (f(t), \phi)_{\omega_\gamma,\Lambda}, \\ \quad \forall \phi \in V_N(\Lambda),\ 0 < t \le T, \\ \partial_t u_N(x,0) = \hat{P}_{N,\beta,\gamma,\Lambda} U_1(x) \quad \text{or} \quad \hat{P}^1_{N,\beta,\gamma,\Lambda} U_1(x), \quad u_N(x,0) = \hat{P}^1_{N,\beta,\gamma,\Lambda} U_0(x),\ x \in \Lambda. \end{cases} \tag{4.19}$$

We now deal with the convergence of the spectral scheme (4.19). Let $U_N = \hat{P}^1_{N,\beta,\gamma,\Lambda} U$. Thanks to (3.11), we obtain from (4.18) that

$$\begin{aligned} &(\partial_t^2 U_N(t), \phi)_{\omega_\gamma,\Lambda} + (\partial_x U_N(t), \partial_x\phi)_{\omega_\gamma,\Lambda} - 2\gamma(\partial_x U_N(t), x(1+x^2)^{-1}\phi)_{\omega_\gamma,\Lambda} \\ &\quad + (\sin U_N(t), \phi)_{\omega_\gamma,\Lambda} + \sum_{j=1}^{4} G_j(t,\phi) = (f(t), \phi)_{\omega_\gamma,\Lambda}, \quad \forall \phi \in V_N(\Lambda),\ 0 < t \le T, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} G_1(t,\phi) &= (\partial_t^2 U(t) - \partial_t^2 U_N(t), \phi)_{\omega_\gamma,\Lambda}, \\ G_2(t,\phi) &= (U_N(t) - U(t), \phi)_{\omega_\gamma,\Lambda}, \\ G_3(t,\phi) &= -2\gamma(\partial_x U(t) - \partial_x U_N(t), x(1+x^2)^{-1}\phi)_{\omega_\gamma,\Lambda}, \\ G_4(t,\phi) &= (\sin U(t) - \sin U_N(t), \phi)_{\omega_\gamma,\Lambda}. \end{aligned}$$

Further, we set $\tilde{U}_N = u_N - U_N$. Then, by subtracting (4.20) from (4.19), we obtain

$$\begin{cases} (\partial_t^2 \tilde{U}_N(t), \phi)_{\omega_\gamma,\Lambda} + (\partial_x \tilde{U}_N(t), \partial_x\phi)_{\omega_\gamma,\Lambda} = \sum_{j=1}^{2} F_j(t,\phi) + \sum_{j=1}^{4} G_j(t,\phi), \\ \qquad \forall \phi \in V_N(\Lambda),\ 0 < t \le T, \\ \partial_t \tilde{U}_N(x,0) = \hat{P}_{N,\beta,\gamma,\Lambda} U_1(x) - \hat{P}^1_{N,\beta,\gamma,\Lambda} U_1(x) \quad \text{or} \quad 0, \quad \tilde{U}_N(x,0) = 0,\ x \in \Lambda, \end{cases} \tag{4.21}$$

where

$$\begin{aligned} F_1(t,\phi) &= -2\left(\cos\left(u_N(t) + \frac{1}{2}\tilde{U}_N(t)\right)\sin\left(\frac{1}{2}\tilde{U}_N(t)\right), \phi\right)_{\omega_\gamma,\Lambda}, \\ F_2(t,\phi) &= 2\gamma(x(1+x^2)^{-1}\partial_x \tilde{U}_N(t), \phi)_{\omega_\gamma,\Lambda}. \end{aligned}$$

Taking $\phi = 2\partial_t \tilde{U}_N(t)$ in (4.21), we deduce that

$$\partial_t(\|\partial_t \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda} + \|\partial_x \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda}) = 2\sum_{j=1}^{2} F_j(t, \partial_t \tilde{U}_N(t)) + 2\sum_{j=1}^{4} G_j(t, \partial_t \tilde{U}_N(t)). \tag{4.22}$$

We next estimate the right side of (4.22). Obviously,

$$|F_1(t, \partial_t \tilde{U}_N(t))| \le \frac{1}{2}(\|\partial_t \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda} + \|\tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda}), \tag{4.23}$$

$$|F_2(t, \partial_t \tilde{U}_N(t))| \le \frac{1}{2}|\gamma|(\|\partial_t \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda} + \|\partial_x \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda}). \tag{4.24}$$

By virtue of (3.12), we derive that

$$|G_1(t, \partial_t \tilde{U}_N(t))| \le \frac{1}{16}\|\partial_t \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda} + c(\beta^2 N)^{1-r}\left\|(1+x^2)^{-\frac{\gamma}{2}}\partial_t^2 U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)}, \tag{4.25}$$

$$|G_j(t, \partial_t \tilde{U}_N(t))| \le \frac{1}{16}\|\partial_t \tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda} + c(\beta^2 N)^{1-r}\left\|(1+x^2)^{-\frac{\gamma}{2}}U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)}, \quad j = 2,3,4. \tag{4.26}$$

Moreover, since $\tilde{U}_N(x,0)=0$, we use the Hölder inequality to obtain

$$\|\tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda}=\int_\Lambda\left(\int_0^t \partial_s\tilde{U}_N(x,s)ds\right)^2\omega_\gamma(x)dx\le t\int_0^t\|\partial_s\tilde{U}_N(s)\|^2_{\omega_\gamma,\Lambda}ds. \tag{4.27}$$

On the other hand, if $\partial_t u_N(x,0)=\hat{P}^1_{N,\beta,\gamma,\Lambda}U_1(x)$, then $\partial_t\tilde{U}_N(x,0)=0$. If $\partial_t u_N(x,0)=\hat{P}_{N,\beta,\gamma,\Lambda}U_1(x)$, then (3.10) implies

$$\|\partial_t\tilde{U}_N(0)\|^2_{\omega_\gamma,\Lambda}\le c(\beta^2N)^{1-r}\left\|(1+x^2)^{-\frac{\gamma}{2}}U_1\right\|^2_{H^{r-1}_{A,\beta}(\Lambda)}. \tag{4.28}$$

For describing the error of the numerical solution, we set

$$E(\tilde{U}_N,t)=\|\partial_t\tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda}+\|\partial_x\tilde{U}_N(t)\|^2_{\omega_\gamma,\Lambda},$$
$$R(U,t)=\left\|(1+x^2)^{-\frac{\gamma}{2}}\partial_t^2U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)}+\left\|(1+x^2)^{-\frac{\gamma}{2}}U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)}.$$

Besides, if $\partial_t u_N(x,0)=\hat{P}^1_{N,\beta,\gamma,\Lambda}U_1(x)$, then we put $\rho(U_1)=0$. If $\partial_t u_N(x,0)=\hat{P}_{N,\beta,\gamma,\Lambda}U_1(x)$, then we put $\rho(U_1)=\|(1+x^2)^{-\frac{\gamma}{2}}U_1\|^2_{H^{r-1}_{A,\beta}(\Lambda)}$.

By substituting (4.23)–(4.27) into (4.22), we obtain

$$\partial_tE(\tilde{U}_N,t)\le\left(|\gamma|+\frac{3}{2}\right)E(\tilde{U}_N,t)+t\int_0^tE(\tilde{U}_N,s)ds+c(\beta^2N)^{1-r}R(U,t).$$

Integrating the above inequality, we derive that at least, for all $0\le t\le T$,

$$\begin{aligned}E(\tilde{U}_N,t)&\le\left(\frac{1}{2}t^2+|\gamma|+\frac{3}{2}\right)\int_0^tE(\tilde{U}_N(t),s)ds+c(\beta^2N)^{1-r}\left(\int_0^tR(U,s)ds+\rho(U_1)\right)\\&\le\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)\int_0^tE(\tilde{U}_N(t),s)ds+c(\beta^2N)^{1-r}\left(\int_0^tR(U,s)ds+\rho(U_1)\right).\end{aligned}$$

Furthermore, we consider the auxiliary function

$$W(t)=\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)\int_0^tW(s)ds+c(\beta^2N)^{1-r}\left(\int_0^tR(U,s)ds+\rho(U_1)\right). \tag{4.29}$$

Obviously, $E(\tilde{U}_N,t)\le W(t)$. Moreover, (4.29) implies

$$\begin{cases}\partial_t\left(e^{-\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)t}W(t)\right)=c(\beta^2N)^{1-r}e^{-\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)t}R(U,t),\quad 0<t\le T,\\ W(0)=c(\beta^2N)^{1-r}\rho(U_1).\end{cases}$$

Therefore,

$$E(\tilde{U}_N,t)\le W(t)\le c(\beta^2N)^{1-r}e^{\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)t}\left(\int_0^tR(U,s)e^{-\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)s}ds+\rho(U_1)\right). \tag{4.30}$$

This fact with (3.12) leads to

$$\begin{aligned}E(U-u_N,t)\le c(\beta^2N)^{1-r}\Bigg(&e^{\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)t}\left(\int_0^tR(U,s)e^{-\left(\frac{1}{2}T^2+|\gamma|+\frac{3}{2}\right)s}ds+\rho(U_1)\right)\\&+\left\|(1+x^2)^{-\frac{\gamma}{2}}\partial_tU(t)\right\|^2_{H^r_{A,\beta}(\Lambda)}+\left\|(1+x^2)^{-\frac{\gamma}{2}}U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)}\Bigg).\end{aligned} \tag{4.31}$$

We now describe the implementation for the spectral scheme (4.19). Let τ be the mesh size in time t. The fully discrete scheme for solving (4.18) is as follows,

$$\begin{aligned}&2(u_N(t+\tau),\phi)_{\omega_\gamma,\Lambda}+\tau^2(\partial_xu_N(t+\tau),\partial_x\phi)_{\omega_\gamma,\Lambda}-2\gamma\tau^2(\partial_xu_N(t+\tau),x(1+x^2)^{-1}\phi)_{\omega_\gamma,\Lambda}\\&\quad=4(u_N(t),\phi)_{\omega_\gamma,\Lambda}-2(u_N(t-\tau),\phi)_{\omega_\gamma,\Lambda}-\tau^2(\partial_xu_N(t-\tau),\partial_x\phi)_{\omega_\gamma,\Lambda}\\&\qquad+2\gamma\tau^2(\partial_xu_N(t-\tau),x(1+x^2)^{-1}\phi)_{\omega_\gamma,\Lambda}-2\tau^2(\sin u_N(t),\phi)_{\omega_\gamma,\Lambda}+\tau^2(f(t+\tau)+f(t-\tau),\phi)_{\omega_\gamma,\Lambda}.\end{aligned} \tag{4.32}$$

Table 13
The global weighted numerical errors for solution (4.34).

	$N=20$	$N=40$	$N=60$	$N=80$	$N=100$	$N=120$	$N=140$	$N=160$
$\tau=10^{-1}$	7.84E−3	6.14E−3	6.14E−3	6.14E−3	6.14E−3	6.17E−3	6.17E−3	6.17E−3
$\tau=10^{-2}$	3.18E−3	1.35E−4	5.50E−5	5.27E−5	5.26E−5	5.27E−5	5.26E−5	5.26E−5
$\tau=10^{-3}$	2.47E−3	1.14E−4	1.42E−5	2.74E−6	7.79E−7	7.33E−7	7.23E−7	7.27E−7
$\tau=10^{-4}$	6.73E−4	4.76E−5	5.50E−6	8.52E−7	1.66E−7	4.88E−8	1.20E−8	1.03E−8

Table 14
The point-wise numerical errors for solution (4.34).

	$N=20$	$N=40$	$N=60$	$N=80$	$N=100$	$N=120$	$N=140$	$N=160$
$\tau=10^{-1}$	3.95E−3	4.38E−3	4.33E−3	4.19E−3	4.32E−3	4.25E−3	4.25E−3	4.25E−3
$\tau=10^{-2}$	1.37E−3	6.51E−5	4.37E−5	4.36E−5	4.57E−5	4.46E−5	4.45E−5	4.45E−5
$\tau=10^{-3}$	1.34E−3	6.14E−5	6.38E−6	1.08E−6	6.08E−7	5.33E−7	5.13E−7	5.13E−7
$\tau=10^{-4}$	2.99E−4	3.11E−5	4.00E−6	6.73E−6	1.35E−7	3.14E−8	8.06E−9	7.37E−9

Let $\phi_l(x)$ be the same as in the last subsection. We expand the numerical solution as

$$u_N(t,x)=\sum_{l=0}^{N}\hat{u}_l(t)\phi_l(x).$$

Inserting the above expression into (4.32) with $\phi=\phi_k(x)$, we obtain

$$\begin{aligned}&\sum_{l=0}^{N}\Big(2(\phi_l,\phi_k)_{\omega_\gamma,\Lambda}+\tau^2(\partial_x\phi_l,\partial_x\phi_k)_{\omega_\gamma,\Lambda}-2\gamma\tau^2(\partial_x\phi_l,x(1+x^2)^{-1}\phi_k)_{\omega_\gamma,\Lambda}\Big)\hat{u}_l(t+\tau)\\ &\quad=\sum_{l=0}^{N}4(\phi_l,\phi_k)_{\omega_\gamma,\Lambda}\hat{u}_l(t)+\sum_{l=0}^{N}\Big(-2(\phi_l,\phi_k)_{\omega_\gamma,\Lambda}-\tau^2(\partial_x\phi_l,\partial_x\phi_k)_{\omega_\gamma,\Lambda}+2\gamma\tau^2(\partial_x\phi_l,x(1+x^2)^{-1}\phi_k)_{\omega_\gamma,\Lambda}\Big)\\ &\qquad\times\hat{u}_l(t-\tau)-2\tau^2(\sin u_N(t),\phi_k)_{\omega_\gamma,\Lambda}+\tau^2(f(t+\tau)+f(t-\tau),\phi_k)_{\omega_\gamma,\Lambda},\quad 0\le k\le N.\end{aligned}\tag{4.33}$$

We can rewrite the system (4.33) as a compact matrix form. To do this, we introduce the matrices $A=(a_{k,l})_{0\le k,l\le N}$, $B=(b_{k,l})_{0\le k,l\le N}$ and $C=(c_{k,l})_{0\le k,l\le N}$, with the following entries:

$$a_{k,l}=(\phi_l,\phi_k)_{\omega_\gamma,\Lambda},\qquad b_{k,l}=(\partial_x\phi_l,\partial_x\phi_k)_{\omega_\gamma,\Lambda},\qquad c_{k,l}=(\partial_x\phi_l,x(1+x^2)^{-1}\phi_k)_{\omega_\gamma,\Lambda}.$$

Also, let $\hat{\mathbf{u}}(t)=(\hat{u}_0(t),\hat{u}_1(t),\ldots,\hat{u}_N(t))^T$ and $\mathbf{F}=(F_0,F_1,\ldots,F_N)^T$, with

$$F_k=(f(t+\tau)+f(t-\tau),\phi_k)_{\omega_\gamma,\Lambda}-2(\sin u_N(t),\phi_k)_{\omega_\gamma,\Lambda}.$$

Then, system (4.33) becomes

$$(2A+\tau^2B-2\gamma\tau^2C)\hat{\mathbf{u}}(t+\tau)=4A\hat{\mathbf{u}}(t)+(-2A-\tau^2B+2\gamma\tau^2C)\hat{\mathbf{u}}(t-\tau)+\tau^2\mathbf{F}.$$

We use the algorithm (4.33) to solve the sine–Gordon equation (4.16) with $f(x,t)\equiv 0$. We measure the numerical accuracy by the global weighted error $E_N^{\beta,\gamma}(t)=\|U(t)-u_N(t)\|_{N,\beta,\gamma,\Lambda}$ and the point-wise error $E_{N,pw}(t)=\max_{0\le j\le N}|U(\hat{\sigma}^{\beta}_{N,j},t)-u_N(\hat{\sigma}^{\beta}_{N,j},t)|$, respectively.

The first test function describing the collisions of soliton–antisoliton, is given by

$$U(x,t)=4\tan^{-1}\left(\frac{\sinh\left(\frac{at}{\sqrt{1-a^2}}\right)}{a\cosh\left(\frac{x}{\sqrt{1-a^2}}\right)}\right),\quad |a|<1,\ a\neq 0.\tag{4.34}$$

Since this test function decays to zero exponentially as $|x|$ increases, we could use (4.33) with any γ. In actual computation, we take $a=0.5$, $\beta=1$ and $\gamma=0$.

In Table 13, we list the global weighted error $E_N^{1,0}(1)$ with various mode N and step size τ. They demonstrate that the numerical errors decay as N increases and τ decreases. This confirms the theoretical analysis. In Table 14, we list the point-wise errors $E_{N,pw}(1)$, which also indicate the convergence of algorithm (4.33).

The sine–Gordon equation (4.16) with $f(x,t)\equiv 0$ also possesses solutions describing the collisions of soliton–soliton, namely,

$$U(x,t)=4\tan^{-1}\left(\frac{a\sinh\left(\frac{x}{\sqrt{1-a^2}}\right)}{\cosh\left(\frac{at}{\sqrt{1-a^2}}\right)}\right),\quad |a|<1,\ a\neq 0.\tag{4.35}$$

Table 15
The global weighted numerical errors for solution (4.35).

	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$	$N = 120$	$N = 140$	$N = 160$
$\tau = 10^{-1}$	2.93E−3	2.26E−3	2.24E−3	2.24E−3	2.24E−3	2.24E−3	2.24E−3	2.24E−3
$\tau = 10^{-2}$	1.97E−3	3.16E−4	6.25E−5	2.77E−5	2.48E−5	2.47E−5	2.47E−5	2.47E−5
$\tau = 10^{-3}$	1.98E−3	3.16E−4	5.77E−5	1.26E−5	3.18E−6	9.28E−7	3.70E−7	2.65E−7
$\tau = 10^{-4}$	1.98E−3	3.16E−4	5.77E−5	1.26E−5	3.18E−6	8.94E−7	2.74E−7	9.13E−8

Table 16
The point-wise numerical errors for solution (4.35).

	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$	$N = 120$	$N = 140$	$N = 160$
$\tau = 10^{-1}$	1.56E−3	1.56E−3	1.56E−3	1.60E−3	1.59E−3	1.60E−3	1.59E−3	1.60E−3
$\tau = 10^{-2}$	1.47E−3	2.49E−4	4.66E−5	1.74E−5	1.73E−5	1.74E−5	1.73E−5	1.74E−5
$\tau = 10^{-3}$	1.48E−3	2.49E−4	4.67E−5	1.04E−5	2.65E−6	7.53E−7	2.33E−7	1.75E−7
$\tau = 10^{-4}$	1.48E−3	2.49E−4	4.68E−5	1.04E−5	2.65E−6	7.53E−7	2.33E−7	7.74E−8

Table 17
The global weighted numerical errors for solution (4.36).

	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$	$N = 120$	$N = 140$	$N = 160$
$\tau = 10^{-1}$	2.92E−3	1.91E−3	1.89E−3	1.89E−3	1.89E−3	1.89E−3	1.89E−3	1.89E−3
$\tau = 10^{-2}$	2.32E−3	2.35E−4	4.22E−5	2.15E−5	2.03E−5	2.02E−5	2.02E−5	2.01E−5
$\tau = 10^{-3}$	2.33E−3	2.35E−4	3.72E−5	7.55E−6	1.83E−6	5.37E−7	2.52E−7	2.08E−7
$\tau = 10^{-4}$	2.33E−3	2.35E−4	3.72E−5	7.55E−6	1.82E−6	4.97E−7	1.49E−7	4.87E−8

Table 18
The point-wise numerical errors for solution (4.36).

	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$	$N = 120$	$N = 140$	$N = 160$
$\tau = 10^{-1}$	1.64E−3	1.64E−3	1.73E−3	1.70E−3	1.72E−3	1.74E−3	1.70E−3	1.72E−3
$\tau = 10^{-2}$	1.72E−3	1.91E−4	3.10E−5	1.81E−5	1.86E−5	1.86E−5	1.82E−5	1.86E−5
$\tau = 10^{-3}$	1.74E−3	1.92E−4	3.11E−5	6.40E−6	1.55E−6	4.27E−7	1.83E−7	1.87E−7
$\tau = 10^{-4}$	1.74E−3	1.92E−4	3.11E−5	6.40E−6	1.55E−6	4.27E−7	1.29E−7	4.20E−8

Here, $U(x,t) \to 2\pi\,\mathrm{sgn}(a)$ as $x \to \infty$, and $U(x,t) \to -2\pi\,\mathrm{sgn}(a)$ as $x \to -\infty$. In this case, we make the transformation

$$U(x,t) = V(x,t) + 4\tan^{-1}(\sinh(x)).$$

Inserting the above expression into (4.16), we obtain the reformed equation for the unknown function $V(x,t)$, with the boundary condition $V(x,t) \to 0$ as $|x| \to \infty$. We use the spectral method similar to (4.19), to obtain the numerical solution $v_N(x,t)$. Finally, the numerical solution of the original problem is given by $u_N(x,t) = v_N(x,t) + 4\tan^{-1}(\sinh(x))$.

In Tables 15 and 16, we list the global weighted errors and the point-wise errors at $t = 1$, with $a = 0.5$, $\beta = 1$ and $\gamma = 0$, vs. various modes N and step sizes τ, respectively. They indicate that the numerical errors decay as N increases and τ decreases.

The third kind of solutions of the sine–Gordon equation (4.16) with $f(x,t) \equiv 0$ describe the kink solitons, namely,

$$U(x,t) = 4\tan^{-1}\left(e^{\frac{\eta(x-at)}{\sqrt{1-a^2}}}\right), \quad |a| < 1, \ \eta = \pm 1. \tag{4.36}$$

For $\eta = 1$, the solutions $U(x,t) \to 2\pi$ as $x \to \infty$, and $U(x,t) \to 0$ as $x \to -\infty$. If $\eta = -1$, then the solutions $U(x,t) \to 0$ as $x \to \infty$, and $U(x,t) \to 2\pi$ as $x \to -\infty$. In this case, we make the transformation

$$U(x,t) = V(x,t) + 4\tan^{-1}(e^{\eta x}).$$

Inserting the above expression into (4.16), we obtain the reformed equation for the unknown function $V(x,t)$, with the boundary condition $V(x,t) \to 0$ as $|x| \to \infty$. We use the spectral method similar to (4.19), to obtain the numerical solution $v_N(x,t)$. Finally, the numerical solution of original problem is given by $u_N(x,t) = v_N(x,t) + 4\tan^{-1}(e^{\eta x})$.

In Tables 17 and 18, we list the global weighted errors and the point-wise errors at $t = 1$, with $a = 0.5$, $\eta = 1$, $\beta = 1$ and $\gamma = 0$, vs. various modes N and step sizes τ, respectively. They demonstrate again that the numerical errors decay as N increases and τ decreases.

4.3. Other problems

The solutions of different practical problems possess different asymptotic behaviors. If the solution $U \sim |x|^\alpha$ for large $|x|$, then $U \in L^2_{|x|^\mu}(\Lambda)$ for any $\mu < -2\alpha - 1$. Such a solution could be approximated by the generalized Hermite approximation with the weight function $\omega_\gamma(x) = (1+x^2)^{-\gamma}$, $\gamma \geq -\frac{1}{2}\mu > \alpha + \frac{1}{2}$.

We first consider the Harry-Dym equation:

$$\begin{cases} \partial_t(U^2(x,t)) + 2\partial_x^3\left(\dfrac{1}{U(x,t)}\right) = 0, & x \in \Lambda, t > 0, \\ U(x,t) \to 1, & \text{as } |x| \to \infty, t > 0, \\ \partial_x^m U(x,t) \to 0, & \text{as } |x| \to \infty, t > 0, m = 1,2,3, \\ U(x,0) = U_0(x), & x \in \Lambda, \end{cases}$$

where $U_0(x)$ is a continuous function and $U_0(x) \to 1$, as $|x| \to \infty$. Since $U(x,t) \sim 1$ at the infinity, the solution is in the space $L^2_{|x|^\mu}(\Lambda)$, $\mu < -1$. Thus, we could use the generalized Hermite orthogonal approximation with $\gamma > \frac{1}{2}$.

Next, let $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ and the m-dimensional infinite channel $\Omega_m = \{\mathbf{x} \mid -1 < x_i < 1 \text{ for } 1 \le i \le m-1, \text{ and } -\infty < x_m < \infty\}$. The velocity $\mathbf{U}(\mathbf{x},t) = (U_1(\mathbf{x},t), U_2(\mathbf{x},t), \ldots, U_m(\mathbf{x},t))^T$. The pressure is denoted by $P(\mathbf{x},t)$. ν is the kinetic viscosity. We consider the incompressible fluid flow in the channel:

$$\begin{cases} \partial_t \mathbf{U}(\mathbf{x},t) + (\mathbf{U}(\mathbf{x},t)\cdot\nabla)\mathbf{U}(\mathbf{x},t) - \nu\triangle\mathbf{U}(\mathbf{x},t) + \nabla P(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t), & \mathbf{x} \in \Omega_m,\ t > 0, \\ \nabla\cdot\mathbf{U}(\mathbf{x},t) = 0, & \mathbf{x} \in \bar{\Omega}_m,\ t \ge 0, \\ \mathbf{U}(\mathbf{x},0) = \mathbf{U}_0(\mathbf{x}), & \mathbf{x} \in \bar{\Omega}_m. \end{cases}$$

According to the mechanical principle, we impose the following boundary condition at the infinity as usual,

$$\mathbf{U} \sim \frac{1}{\ln|x_m|} \quad \text{for } m = 2, \qquad \mathbf{U} \sim \frac{1}{|x_m|} \quad \text{for } m = 3.$$

Thus, U is in the space $(L^2_{|x|^\mu}(\Lambda))^m$ where $\mu < -1$ for $m = 2$, and $\mu < 1$ for $m = 3$. Therefore, we could use the generalized Hermite orthogonal approximation in the x_m-direction, where $\gamma > \frac{1}{2}$ for $m = 2$ and $\gamma > -\frac{1}{2}$ for $m = 3$. Meanwhile, we use the Legendre orthogonal approximation in the other directions.

Third, we consider the Klein–Gordon equation:

$$\partial_t^2 U(x,t) + \frac{1}{2}U(x,t) - U^3(x,t) - \partial_x^2 U(x,t) = 0,$$

which has the bell soliton solution as

$$U(x,t) = -\operatorname{sech}\left(\frac{x - at}{\sqrt{2 - 2a^2}}\right), \quad |a| < 1.$$

In this case, $U(x,t)$ decays exponentially, and so belongs to the space $(L^2_{|x|^\mu}(\Lambda))^m$, μ being an arbitrary real number. Consequently, we could adopt the generalized Hermite orthogonal approximation with any real number γ.

The solutions of many practical problems, such as the heteroclinic solutions in biology and the kink solitons in quantum mechanics, behave differently at the different endpoints of the infinite interval. In those cases, it seems better to use the generalized Hermite orthogonal approximation with the weight function $(1 + \frac{2}{\pi}\arctan x)^{-2\alpha}(1 - \frac{2}{\pi}\arctan x)^{-2\gamma}$, α and γ being certain real numbers. By adjusting the parameters α and γ suitably, it may simulate different asymptotic behaviors of approximated functions at the different endpoints.

5. Concluding remarks

In this paper, we introduced the new orthogonal system with the weight function $(1 + x^2)^{-\gamma}$, γ being any real number. By adjusting the parameter γ suitably, such system may simulate the asymptotic behaviors of approximated functions reasonably. We established the basic results on the corresponding orthogonal approximation and interpolation, which play important roles in the spectral and pseudospectral methods for various problems defined on the whole line and the related multiple-dimensional unbounded domains. As examples of applications, we provided the spectral schemes for a linear problem and the sine–Gordon equation. We proved their spectral accuracy in space. The numerical results demonstrated the efficiency of the suggested algorithms, and coincided well with the analysis. In particular, the proposed approaches not only possess the spectral accuracy in the global weighted norms, but also possess the small point-wise numerical errors.

It is noted that Guo and Shen [13], and Guo and Zhang [14] provided the Jacobi irrational spectral method and the generalized Laguerre spectral method for the half line, while Guo and Yi [12], and Yi and Guo [15] considered the generalized Jacobi rational spectral methods for infinite intervals. We also refer the reader to the recent review paper of Guo [16].

Acknowledgments

The work of the first author is supported in part by NSF of China N.11171227, Research Fund for Young Teachers of Jiangsu Normal University N.11XLR27, and Priority Academic Program Development of Jiangsu Higher Education Institutions.

The work of the second author is supported in part by NSF of China N.11171227, Fund for Doctoral Authority of China N.20123127110001, Fund for E-institute of Shanghai Universities N.E03004, and Leading Academic Discipline Project of Shanghai Municipal Education Commission N.J50101.

References

[1] Ben-yu Guo, Error estimation of Hermite spectral method for nonlinear partial differential equations, Math. Comp. 68 (1999) 1067–1078.
[2] Ben-yu Guo, Cheng-long Xu, Hermite pseudospectral method for nonlinear partial differential equations, M2AN Math. Model. Numer. Anal. 34 (2000) 859–872.
[3] J.A.C. Weideman, The eigenvalues of Hermite and rational differentiation matrices, Numer. Math. 61 (1992) 409–431.
[4] D. Funaro, O. Kavian, Approximation of some diffusion evolution equations in unbounded domains by Hermite function, Math. Comp. 57 (1999) 597–619.
[5] C.M. Fok Johnson, Ben-yu Guo, Tao Tang, Hermite spectral-finite difference method for the Fokker–Planck equation, Math. Comp. 71 (2002) 1497–1528.
[6] Ben-yu Guo, Jie Shen, Cheng-long Xu, Spectral and pseudospectral approximations using Hermite functions: application to the Dirac equation, Adv. Comput. Math. 19 (2003) 35–55.
[7] J.P. Boyd, The rate of convergence of Hermite function series, Math. Comp. 35 (1980) 1309–1316.
[8] J.P. Boyd, The asymptotic coefficients of Hermite series, J. Comput. Phys. 54 (1984) 382–410.
[9] He-ping Ma, Wei-wei Sun, Tao Tang, Hermite spectral methods with a time-dependent scaling for parabolic equations in unbounded domains, SIAM J. Numer. Anal. 43 (2005) 58–75.
[10] He-ping Ma, Ting-gang Zhao, A stabilized Hermite spectral method for second-order differential equations in unbounded domain, Numer. Methods Partial Differential Equations 23 (2007) 968–983.
[11] Xin-min Xiang, Zhong-qing Wang, Generalized Hermite spectral method and its applications to problems in unbounded domains, SIAM J. Numer. Anal. 48 (2010) 1231–1253.
[12] Ben-yu Guo, Yong-gang Yi, Generalized Jacobi rational spectral method and its applications, J. Sci. Comput. 43 (2010) 201–238.
[13] Ben-yu Guo, Jie Shen, Irrational approximations and their applications to partial differential equations in exterior domains, Adv. Comput. Math. 28 (2008) 237–267.
[14] Ben-yu Guo, Chao Zhang, Spectral method for high order problems with proper simulations of asymptotic behaviors at infinity, J. Comput. Appl. Math. 237 (2013) 269–294.
[15] Yong-gang Yi, Ben-yu Guo, Generalized Jacobi rational spectral method on the half line, Adv. Comput. Math. 37 (2012) 1–37.
[16] Ben-yu Guo, Some progress in spectral methods, Sci. China Math. (2013) http://dx.doi.org/10.1007/s11425-013-4660-7.