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Generalized Hermite Spectral Method Matching Different Algebraic Decay at infinities

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Abstract In this paper, we propose a new generalized Hermite spectral method. We introduce an orthogonal family of new generalized Hermite functions, with the weight function $(1 + \frac{2}{\pi}\arctan x)^{\alpha}(1 - \frac{2}{\pi}\arctan x)^{\gamma}$, α and γ being arbitrary real numbers. The basic results on the corresponding orthogonal approximation and interpolation are established. As examples of applications, we provide the spectral schemes for a linear problem and the Fisher equation, which possess the spectral accuracy in space and match the different algebraic decay at infinities reasonably. Numerical results demonstrate their high efficiency and coincide well with the analysis.

Keywords New generalized Hermite orthogonal approximation and interpolation · Spectral method on the whole line · Matching different algebraic decay at infinities

Mathematics Subject Classification 41A30 · 65L60 · 65M70 · 34B40 · 35L70

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1 Introduction

In many practical cases occurring in quantum mechanics, biology, financial mathematics and other fields, we have to consider numerical simulations of various differential equations defined on unbounded domains. For problems defined on the whole line and certain related unbounded domains, it is natural to use the Hermite orthogonal approximation and the Hermite–Gauss interpolation. Guo [7], and Guo and Xu [8] developed the spectral and pseudospectral methods for nonlinear partial differential equations, by using the Hermite polynomials which are mutually orthogonal with the weight function e^{-x^2}. Weideman [19] presented some results on their implementations. The above approaches are available even if the solutions behave like $e^{\theta x^2}$, $\theta < \frac{1}{2}$, for large $|x|$. Whereas, the small global e^{-x^2}-weighted errors do not imply the small point-wise errors automatically. Meanwhile, Funaro and Kavian [4] considered the spectral method for linear parabolic equations by using an orthogonal system with the weight function $e^{\frac{1}{4}x^2}$. Fok et al. [3] proposed the orthogonal approximation with the weight function $e^{\theta x^2}$, $\theta > 0$. They also used it coupled with finite difference approximation, to the simplified Fokker–Planck equation. This approach is only suitable for problems with solutions decaying faster than $e^{-\frac{1}{2}\theta x^2}$ as $|x|$ increases. On the other hand, Guo et al. [9] provided the spectral and pseudospectral methods using the Hermite functions which are mutually orthogonal with the weight function 1. They are appropriate for problems with solutions behaving like $(1+x^2)^{-\frac{1}{2}\theta}$, $\theta > 1$, for large $|x|$, and were applied to numerical simulation of the Dirac equation. We also refer to the work of Boyd [1,2], Ma et al. [15], Ma and Zhao [16], and Xiang and Wang [20], and the review papers of Guo [14], and Shen and Wang [17]. Recently, Zhang and Guo [22] provided the spectral method using the generalized Hermite functions which are mutually orthogonal with the weight function $(1+x^2)^{-\gamma}$, γ being arbitrary real number. If the solutions of underlying problems behave like $(1+x^2)^{\frac{1}{2}\alpha}$, $\alpha < \gamma - \dfrac{1}{2}$, for large $|x|$, then the obtained numerical solutions fit such behaviors reasonably.

The solutions of many practical problems, such as the heteroclinic solutions in biology and the kink solitons in quantum mechanics, behave differently at infinities. For instance, the solutions might decay at certain algebraic rates as $x \to -\infty$, but decay at other algebraic rates as $x \to \infty$. In these cases, the method given in [22] is no longer the most appropriate. Thus, we need certain new orthogonal approximation with the weight function behaving differently at infinities. As we know, Guo et al. [10,11] developed the generalized Jacobi approximation with the weight function $(1-x)^{\alpha}(1+x)^{\beta}$, α and β being any real numbers, which leads to the new Jacobi spectral and pseudospectral methods for differential equations defined on the finite interval, fitting the different behaviors of approximated functions at the endpoints $x = \pm 1$. Sun and Guo [18] considered the generalized Jacobi approximation in multiple dimensions with its applications. Meanwhile, Guo and Yi [12] and Yi and Guo [21] proposed the generalized Jacobi irrational spectral methods for differential equations defined on the whole line and the half line, simulating the different behaviors at infinities. Furthermore, Guo and Zhang [13] studied the generalized Laguerre approximation with the weight function $x^{\alpha}(1+x)^{\gamma}$, α and γ being any real numbers, which induce the efficient spectral and pseudospectral methods for various problems defined on the half line, simulating the properties of approximated functions at $x = 0, \infty$. However, so far, there has not been any existing results on the Hermite approximation matching different asymptotic behaviors at infinities.

In this paper, we propose a new Hermite spectral method on the whole line. For this purpose, we introduce a family of new generalized Hermite functions, which are mutually orthogonal with the weight function $(1+\frac{2}{\pi}\arctan x)^{-2\alpha}(1-\frac{2}{\pi}\arctan x)^{-2\gamma}$, α and γ being any real numbers. By adjusting the parameters α and γ suitably, they simulate different asymptotic behaviors of approximated functions at infinities properly. We establish the basic results on the corresponding generalized Hermite orthogonal approximation, which play an important role in the related spectral method for various problems with different kinds of asymptotic behaviors at infinities. Then, we study the related generalized Hermite–Gauss interpolation, serving as the mathematical foundation of the new Hermite pseudospectral method. As examples of applications, we provide the spectral schemes for a linear problem and the Fisher equation in biology, and prove their spectral accuracy in space. The numerical results indicate the high efficiency of the suggested algorithms, and coincide well with the analysis.

This paper is organized as follows. The next section is for preliminaries. In Sect. 3, we study the new generalized Hermite orthogonal approximation and the new generalized Hermite–Gauss interpolation. In Sect. 4, we provide the spectral schemes for two model problems, and present some numerical results. The final section is for concluding remarks.

2 Preliminaries

In this section, we recall some results on the Hermite approximation.

Let $\Lambda = \{\, x \mid -\infty < x < \infty\}$ and $\chi(x)$ be a certain weight function. For integer $r \geq 0$, we define the weighted Hilbert space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(\cdot,\cdot)_{r,\chi,\Lambda}$, the semi-norm $|\cdot|_{r,\chi,\Lambda}$ and the norm $||\cdot||_{r,\chi,\Lambda}$. In particular, the inner product and the norm of the space $L^2_\chi(\Lambda)$ are denoted by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$, respectively. We omit the subscript χ in notations whenever $\chi(x)\equiv 1$. For simplicity, we denote $\dfrac{d^k v}{dx^k}$ by $\partial_x^k v$, etc.

Let $H_l(x)$ be the standard Hermite polynomial of degree l. For any $\beta > 0$, the scaled Hermite functions are defined by

$$H^\beta_l(x) = \frac{1}{\sqrt{2^l l!}} H_l(\beta x) e^{-\frac{1}{2}\beta^2 x^2}, \quad l \geq 0.$$

They are the eigenfunctions of the following singular Sturm-Liouville problem,

$$e^{\frac{1}{2}\beta^2 x^2}\partial_x(e^{-\beta^2 x^2}\partial_x(e^{\frac{1}{2}\beta^2 x^2} v(x))) + \lambda^\beta_l v(x) = 0, \quad \lambda^\beta_l = 2\beta^2 l, \quad l \geq 0. \tag{2.1}$$

Let $\delta_{l,m}$ be the Kronecker symbol. The set of all $H^\beta_l(x)$ is a complete $L^2(\Lambda)-$orthogonal system, namely,

$$\int_\Lambda H^\beta_l(x) H^\beta_m(x) dx = \frac{\sqrt{\pi}}{\beta}\delta_{l,m}. \tag{2.2}$$

For any $v \in L^2(\Lambda)$, we have

$$v(x) = \sum_{l=0}^{\infty} v^\beta_l H^\beta_l(x), \tag{2.3}$$

with

$$v^\beta_l = \frac{\beta}{\sqrt{\pi}}\int_\Lambda v(x) H^\beta_l(x) dx.$$

For any positive integer N, we set

$$\mathcal{Q}^\beta_N(\Lambda) = \mathrm{span}\{\, H^\beta_0(x), H^\beta_1(x), \cdots, H^\beta_N(x)\,\}.$$

The $L^2(\Lambda)$-orthogonal projection $P_{N,\beta,\Lambda} : L^2(\Lambda) \to \mathcal{Q}_N^\beta(\Lambda)$ is defined by

$$(P_{N,\beta,\Lambda}v - v, \phi) = 0, \quad \forall \phi \in \mathcal{Q}_N^\beta(\Lambda). \tag{2.4}$$

For any integer $r \geq 0$, we define the space

$$H^r_{A,\beta}(\Lambda) = \{ v \mid \|v\|_{H^r_{A,\beta}(\Lambda)} < \infty\},$$

equipped with the norm

$$\|v\|_{H^r_{A,\beta}(\Lambda)} = \left(\sum_{k=0}^{r} \|(\beta^4 x^2 + \beta^2)^{\frac{r-k}{2}} \partial_x^k v\|^2_\Lambda \right)^{\frac{1}{2}}.$$

Throughout this paper, we denote by c a generic positive constant independent of any function, N and β. By virtue of Theorem 2.1 of [20], we have the following result.

Lemma 2.1 *If $v \in H^r_{A,\beta}(\Lambda)$ and integers $0 \leq k \leq r$, then*

$$\|P_{N,\beta,\Lambda}v - v\|_{k,\Lambda} \leq c(\beta^2 N)^{\frac{k-r}{2}} \|v\|_{H^r_{A,\beta}(\Lambda)}. \tag{2.5}$$

The above result with $\beta = 1$ was first given by Guo et al. see Theorem 2.3 of [9].

Next, let $\sigma_{N,j}$ and $\omega_{N,j}$ be the nodes and the weights of the standard Hermite–Gauss interpolation, $0 \leq j \leq N$, (cf. [8]). We take the nodes and the weights of the scaled Hermite–Gauss interpolation as follows (cf. [22]),

$$\sigma^\beta_{N,j} = \frac{\sigma_{N,j}}{\beta}, \quad \omega^\beta_{N,j} = \frac{1}{\beta}\omega_{N,j} e^{\sigma^2_{N,j}}, \quad 0 \leq j \leq N.$$

It was shown by (2.6) of [22] that for any integer $m \geq 0$,

$$\sum_{j=0}^{N} \phi(\sigma^\beta_{N,j})\psi(\sigma^\beta_{N,j})\omega^\beta_{N,j} = (\phi, \psi)_\Lambda, \quad \forall \phi \in \mathcal{Q}^\beta_m(\Lambda), \psi \in \mathcal{Q}^\beta_{2N+1-m}(\Lambda). \tag{2.6}$$

For any $v \in C(\Lambda)$, the scaled Hermite–Gauss interpolation $I_{N,\beta,\Lambda}v \in \mathcal{Q}^\beta_N(\Lambda)$ is determined uniquely by

$$I_{N,\beta,\Lambda}v(\sigma^\beta_{N,j}) = v(\sigma^\beta_{N,j}), \quad 0 \leq j \leq N. \tag{2.7}$$

According to Lemma 2.2 of [22], we have the following result.

Lemma 2.2 *If $v \in H^r_{A,\beta}(\Lambda)$, integers $r \geq 1$ and $0 \leq k \leq r$, then*

$$\|I_{N,\beta,\Lambda}v - v\|_{k,\Lambda} \leq c(\beta^k + 1)(\beta^2 N)^{\frac{1}{3}+\frac{k-r}{2}} \|v\|_{H^r_{A,\beta}(\Lambda)}. \tag{2.8}$$

3 New Generalized Hermite Orthogonal Approximation and Interpolation

In this section, we develop the new generalized Hermite orthogonal approximation and interpolation, which match different asymptotic behaviors of considered functions at infinities reasonably.

3.1 New Generalized Hermite Functions

For any real numbers α and γ, the new generalized Hermite functions are defined by

$$\hat{H}_l^{\alpha,\beta,\gamma}(x) = F_{\alpha,\gamma}(x) H_l^{\beta}(x), \quad \beta > 0, \quad l \geq 0,$$

where

$$F_{\alpha,\gamma}(x) = \left(1 + \frac{2}{\pi} \arctan x\right)^{\alpha} \left(1 - \frac{2}{\pi} \arctan x\right)^{\gamma}.$$

With the aid of the L'Hospital rule, we obtain that

$$\lim_{x \to +\infty} \frac{1 - \frac{2}{\pi} \arctan x}{(1+x)^{-1}} = \frac{2}{\pi} \lim_{x \to +\infty} \frac{(1+x^2)^{-1}}{(1+x)^{-2}} = \frac{2}{\pi},$$
$$\lim_{x \to -\infty} \frac{1 + \frac{2}{\pi} \arctan x}{(1+x)^{-1}} = -\frac{2}{\pi} \lim_{x \to -\infty} \frac{(1+x^2)^{-1}}{(1+x)^{-2}} = -\frac{2}{\pi}.$$

In other words,

$$\begin{aligned} &1 + \frac{2}{\pi} \arctan x \to 2, \quad 1 - \frac{2}{\pi} \arctan x \sim \frac{2}{\pi(1+x)}, \quad \text{as } x \to +\infty, \\ &1 - \frac{2}{\pi} \arctan x \to 2, \quad 1 + \frac{2}{\pi} \arctan x \sim -\frac{2}{\pi(1+x)}, \quad \text{as } x \to -\infty. \end{aligned} \tag{3.1}$$

On the other hand, the scaled Hermite functions $H_l^{\beta} \in L^2(\Lambda)$ and so $H_l^{\beta}(x) = o((1+|x|)^{-\frac{1}{2}})$ as $|x| \to \infty$. Therefore,

$$\hat{H}_l^{\alpha,\beta,\gamma}(x) = \begin{cases} o((1+|x|)^{-\gamma-\frac{1}{2}}), & \text{as } x \to +\infty, \\ o((1+|x|)^{-\alpha-\frac{1}{2}}), & \text{as } x \to -\infty. \end{cases}$$

Moreover, thanks to (2.1), the functions $\hat{H}_l^{\alpha,\beta,\gamma}(x)$ are the l-th eigenfunctions of the following Sturm-Liouville problem,

$$e^{\frac{1}{2}\beta^2 x^2} \partial_x \left(e^{-\beta^2 x^2} \partial_x \left(F_{\alpha,\gamma}^{-1}(x) e^{\frac{1}{2}\beta^2 x^2} v(x)\right)\right) + \lambda_l^{\beta}, F_{\alpha,\gamma}^{-1}(x) v(x) = 0, \quad l \geq 0. \tag{3.2}$$

Now, let the weight function

$$\omega_{\alpha,\gamma}(x) = F_{\alpha,\gamma}^{-2}(x) = \left(1 + \frac{2}{\pi} \arctan x\right)^{-2\alpha} \left(1 - \frac{2}{\pi} \arctan x\right)^{-2\gamma}.$$

Thanks to (3.1), we have that

$$\begin{aligned} \omega_{\alpha,\gamma}(x) &= \mathcal{O}((1+|x|)^{2\gamma}), \quad \text{as } x \to +\infty, \\ \omega_{\alpha,\gamma}(x) &= \mathcal{O}((1+|x|)^{2\alpha}), \quad \text{as } x \to -\infty. \end{aligned} \tag{3.3}$$

By virtue of (2.2), the set of all $\hat{H}_l^{\alpha,\beta,\gamma}(x)$ is a complete $L^2_{\omega_{\alpha,\gamma}}(\Lambda)$-orthogonal system, namely,

$$\int_{\Lambda} \hat{H}_l^{\alpha,\beta,\gamma}(x) \hat{H}_m^{\alpha,\beta,\gamma}(x) \omega_{\alpha,\gamma}(x) dx = \frac{\sqrt{\pi}}{\beta} \delta_{l,m}. \tag{3.4}$$

Thus, for any $v \in L^2_{\omega_{\alpha,\gamma}}(\Lambda)$, we have

$$v(x) = \sum_{l=0}^{\infty} \hat{v}_l^{\alpha,\beta,\gamma} \hat{H}_l^{\alpha,\beta,\gamma}(x), \tag{3.5}$$

with

$$\hat{v}_l^{\alpha,\beta,\gamma} = \frac{\beta}{\sqrt{\pi}} \int_\Lambda v(x) \hat{H}_l^{\alpha,\beta,\gamma}(x) \omega_{\alpha,\gamma}(x) dx. \tag{3.6}$$

3.2 New Generalized Hermite Orthogonal Approximation

Let

$$\hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda) = \text{span}\left\{ \hat{H}_0^{\alpha,\beta,\gamma}(x), \hat{H}_1^{\alpha,\beta,\gamma}(x), \cdots, \hat{H}_N^{\alpha,\beta,\gamma}(x) \right\}.$$

The orthogonal projection $\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} : L^2_{\omega_{\alpha,\gamma}}(\Lambda) \to \hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda)$ is defined by

$$(\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v - v, \phi)_{\omega_{\alpha,\gamma},\Lambda} = 0, \quad \forall \phi \in \hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda),$$

or equivalently,

$$\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v(x) = \sum_{l=0}^N \hat{v}_l^{\alpha,\beta,\gamma} \hat{H}_l^{\alpha,\beta,\gamma}(x). \tag{3.7}$$

Theorem 3.1 *If $v \in H^r_{\omega_{\alpha,\gamma}}(\Lambda)$ and integers $0 \le k \le r$, then*

$$\|\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v - v\|_{k,\omega_{\alpha,\gamma},\Lambda} \le c(\beta^2 N)^{\frac{k-r}{2}} \|F^{-1}_{\alpha,\gamma}(x) v\|_{H^r_{A,\beta}(\Lambda)}. \tag{3.8}$$

Proof Since $v \in H^r_{\omega_{\alpha,\gamma}}(\Lambda) \subset L^2_{\omega_{\alpha,\gamma}}(\Lambda)$, we derive that

$$\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v(x) = \sum_{l=0}^N \hat{v}_l^{\alpha,\beta,\gamma} \hat{H}_l^{\alpha,\beta,\gamma}(x) = F_{\alpha,\gamma}(x) \sum_{l=0}^N \hat{v}_l^{\alpha,\beta,\gamma} H_l^\beta(x).$$

Let $P_{N,\beta,\Lambda} v$ be the same as in (2.4). With the aid of (3.6), it is easy to show that all coefficients $\hat{v}_l^{\alpha,\beta,\gamma}$ are exactly the same as the coefficients of expansion (2.3) for the function $F^{-1}_{\alpha,\gamma}(x) v(x)$. In other words,

$$\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v(x) = F_{\alpha,\gamma}(x) P_{N,\beta,\Lambda}(F^{-1}_{\alpha,\gamma}(x) v(x)).$$

Therefore,

$$\begin{aligned} \|\partial_x^k(\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v - v)\|_{\omega_{\alpha,\gamma},\Lambda} &= \|\partial_x^k(F_{\alpha,\gamma}(P_{N,\beta,\Lambda}(F^{-1}_{\alpha,\gamma} v) - F^{-1}_{\alpha,\gamma} v))\|_{\omega_\gamma,\Lambda} \\ &= \| \sum_{j=0}^k C_k^j \partial_x^{k-j} F_{\alpha,\gamma} \partial_x^j \left(P_{N,\beta,\Lambda}\left(F^{-1}_{\alpha,\gamma} v\right) - F^{-1}_{\alpha,\gamma} v\right) \|_{\omega_{\alpha,\gamma},\Lambda}. \end{aligned} \tag{3.9}$$

Thanks to (3.1), a direct calculation shows

$$|\partial_x^{k-j}(F_{\alpha,\gamma}(x))| \le c\left(1 + \frac{2}{\pi} \arctan x\right)^\alpha \left(1 - \frac{2}{\pi} \arctan x\right)^\gamma, \quad 0 \le j \le k. \tag{3.10}$$

By using (3.10) and (2.5), we obtain from (3.9) that

$$\begin{aligned} \|\partial_x^k(\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} v - v)\|_{\omega_{\alpha,\gamma},\Lambda} &\le c \sum_{j=0}^k C_k^j \|\partial_x^j (P_{N,\beta,\Lambda}(F^{-1}_{\alpha,\gamma} v) - F^{-1}_{\alpha,\gamma} v)\|_\Lambda \\ &\le c(\beta^2 N)^{\frac{k-r}{2}} \|F^{-1}_{\alpha,\gamma}(x) v\|_{H^r_{A,\beta}(\Lambda)}. \end{aligned}$$

This leads to the desired result (3.8). □

In numerical analysis of the related spectral methods, we need the orthogonal approximation in the space $H^1_{\omega_{\alpha,\gamma}}(\Lambda)$. The orthogonal projection $\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda} : H^1_{\omega_{\alpha,\gamma}}(\Lambda) \to \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_N(\Lambda)$ is defined by

$$(\partial_x(\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}v - v), \partial_x\phi)_{\omega_{\alpha,\gamma},\Lambda} + (\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}v - v, \phi)_{\omega_{\alpha,\gamma},\Lambda} = 0, \quad \forall\phi \in \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_N(\Lambda). \tag{3.11}$$

Theorem 3.2 *If $v \in H^r_{\omega_{\alpha,\gamma}}(\Lambda)$ and integer $r \geq 1$, then*

$$\|\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}v - v\|_{1,\omega_{\alpha,\gamma},\Lambda} \leq c(\beta^2 N)^{\frac{1-r}{2}}\|F^{-1}_{\alpha,\gamma}v\|^r_{H_{A,\beta}(\Lambda)}. \tag{3.12}$$

Proof According to the projection theorem, we have

$$\|\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}v - v\|_{1,\omega_{\alpha,\gamma},\Lambda} \leq \|\phi - v\|_{1,\omega_{\alpha,\gamma},\Lambda}, \quad \forall\phi \in \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_N(\Lambda). \tag{3.13}$$

Let $w(x) = F^{-1}_{\alpha,\gamma}(x)v(x)$. Since $v \in H^r_{\omega_{\alpha,\gamma}}(\Lambda)$, we assert $w \in H^r(\Lambda)$. Therefore, we could take $\phi = F_{\alpha,\gamma}P_{N,\beta,\Lambda}w \in \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_N(\Lambda)$ in (3.13). Next, we use (2.5) to verify that

$$\begin{aligned}\|F_{\alpha,\gamma}P_{N,\beta,\Lambda}w - v\|_{\omega_{\alpha,\gamma},\Lambda} &= \|F_{\alpha,\gamma}(P_{N,\beta,\Lambda}w - w)\|_{\omega_{\alpha,\gamma},\Lambda}\\ &= \|P_{N,\beta,\Lambda}w - w\|_{\Lambda}\\ &\leq c(\beta^2 N)^{-\frac{r}{2}}\|w\|_{H^r_{A,\beta}(\Lambda)}\\ &= c(\beta^2 N)^{-\frac{r}{2}}\|F^{-1}_{\alpha,\gamma}v\|_{H^r_{A,\beta}(\Lambda)}.\end{aligned} \tag{3.14}$$

Also, we use (3.10) and (2.5) to deduce that

$$\begin{aligned}\|\partial_x(F_{\alpha,\gamma}P_{N,\beta,\Lambda}w - v)\|_{\omega_{\alpha,\gamma},\Lambda} &= \|\partial_x(F_{\alpha,\gamma}(P_{N,\beta,\Lambda}w - w))\|_{\omega_{\alpha,\gamma},\Lambda}\\ &\leq \|F_{\alpha,\gamma}\partial_x(P_{N,\beta,\Lambda}w - w)\|_{\omega_{\alpha,\gamma},\Lambda}\\ &\quad + \|\partial_x F_{\alpha,\gamma}(P_{N,\beta,\Lambda}w - w)\|_{\omega_{\alpha,\gamma},\Lambda}\\ &\leq \|\partial_x(P_{N,\beta,\Lambda}w - w)\|_{\Lambda} + c\|P_{N,\beta,\Lambda}w - w\|_{\Lambda}\\ &\leq c(\beta^2 N)^{\frac{1-r}{2}}\|F^{-1}_{\alpha,\gamma}v\|_{H^r_{A,\beta}(\Lambda)}.\end{aligned} \tag{3.15}$$

A combination of (3.13)–(3.15) leads to the desired result (3.12). □

3.3 New Generalized Hermite–Gauss Interpolation

In this subsection, we propose the new generalized Hermite–Gauss interpolation. Let $\sigma^\beta_{N,j}$ and $\omega^\beta_{N,j}$ be the same as in Sect. 2. We set

$$\hat{\sigma}^\beta_{N,j} = \sigma^\beta_{N,j}, \quad \hat{\omega}^{\alpha,\beta,\gamma}_{N,j} = F^{-2}_{\alpha,\gamma}\left(\hat{\sigma}^\beta_{N,j}\right)\omega^\beta_{N,j}, \quad 0 \leq j \leq N. \tag{3.16}$$

The related discrete inner product and norm are defined by

$$(u, v)_{N,\alpha,\beta,\gamma,\Lambda} = \sum_{j=0}^{N} u\left(\hat{\sigma}^\beta_{N,j}\right) v\left(\hat{\sigma}^\beta_{N,j}\right)\hat{\omega}^{\alpha,\beta,\gamma}_{N,j}, \quad \|v\|_{N,\alpha,\beta,\gamma,\Lambda} = (v, v)^{\frac{1}{2}}_{N,\alpha,\beta,\gamma,\Lambda}.$$

For any $\phi \in \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_m(\Lambda)$ and $\psi \in \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_{2N+1-m}(\Lambda)$, we have $\phi(x) = F_{\alpha,\gamma}(x)q_\phi(x)$ and $\psi(x) = F_{\alpha,\gamma}(x)q_\psi(x)$, where $q_\phi \in \mathcal{Q}^\beta_m(\Lambda)$ and $q_\psi \in \mathcal{Q}^\beta_{2N+1-m}(\Lambda)$, respectively. Thereby, we use (2.6) to verify that

$$(\phi,\psi)_{\omega_{\alpha,\gamma},\Lambda}=(q_\phi,q_\psi)_\Lambda=\sum_{j=0}^{N}q_\phi\left(\sigma_{N,j}^{\beta}\right)q_\psi\left(\sigma_{N,j}^{\beta}\right)\omega_{N,j}^{\beta}$$
$$=\sum_{j=0}^{N}\phi\left(\hat\sigma_{N,j}^{\beta}\right)\psi\left(\hat\sigma_{N,j}^{\beta}\right)\hat\omega_{N,j}^{\alpha,\beta,\gamma}=(\phi,\psi)_{N,\alpha,\beta,\gamma,\Lambda},$$
$$\forall\phi\in\hat{\mathcal{Q}}_m^{\alpha,\beta,\gamma}(\Lambda),\ \psi\in\hat{\mathcal{Q}}_{2N+1-m}^{\alpha,\beta,\gamma}(\Lambda). \tag{3.17}$$

In particular,

$$\|\phi\|_{\omega_{\alpha,\gamma},\Lambda}=\|\phi\|_{N,\alpha,\beta,\gamma,\Lambda},\quad \forall\,\phi\in\hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda). \tag{3.18}$$

For any $v\in C(\Lambda)$, the new generalized Hermite–Gauss interpolation $\hat I_{N,\alpha,\beta,\gamma,\Lambda}v\in\hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda)$ is determined uniquely by

$$\hat I_{N,\alpha,\beta,\gamma,\Lambda}v\left(\hat\sigma_{N,j}^{\beta}\right)=v(\hat\sigma_{N,j}^{\beta}),\quad 0\le j\le N. \tag{3.19}$$

We now estimate the approximation error of the interpolation $\hat I_{N,\alpha,\beta,\gamma,\Lambda}v$.

Theorem 3.3 *If $v\in H^r_{\omega_{\alpha,\gamma}}(\Lambda)$, $F^{-1}_{\alpha,\gamma}v\in H^r_{A,\beta}(\Lambda)$, integers $r\ge1$ and $0\le k\le r$, then*

$$\|\hat I_{N,\alpha,\beta,\gamma,\Lambda}v-v\|_{k,\omega_{\alpha,\gamma},\Lambda}\le c(\beta^k+1)(\beta^2N)^{\frac13+\frac{k-r}{2}}\|F^{-1}_{\alpha,\gamma}v\|_{H^r_{A,\beta}(\Lambda)}. \tag{3.20}$$

Proof We have from (2.7) and (3.19) that

$$F^{-1}_{\alpha,\gamma}(\hat\sigma_{N,j}^{\beta})\hat I_{N,\alpha,\beta,\gamma,\Lambda}v\left(\hat\sigma_{N,j}^{\beta}\right)=I_{N,\beta,\Lambda}\left(F^{-1}_{\alpha,\gamma}(x)v(x)\right)\Big|_{x=\hat\sigma_{N,j}^{\beta}},\quad 0\le j\le N.$$

Moreover, both of $F^{-1}_{\alpha,\gamma}(x)\hat I_{N,\alpha,\beta,\gamma,\Lambda}v(x)$ and $I_{N,\beta,\Lambda}(F^{-1}_{\alpha,\gamma}(x)v(x))$ belong to the same finite-dimensional set $\mathcal{Q}_N^{\beta}(\Lambda)$. The previous statements imply

$$\hat I_{N,\alpha,\beta,\gamma,\Lambda}v(x)=F_{\alpha,\gamma}(x)I_{N,\beta,\Lambda}\left(F^{-1}_{\alpha,\gamma}(x)v(x)\right).$$

Consequently, we use (3.10) and (2.8) to verify that

$$\begin{aligned}\left\|\partial_x^k(\hat I_{N,\alpha,\beta,\gamma,\Lambda}v-v)\right\|_{\omega_{\alpha,\gamma},\Lambda}&=\left\|\partial_x^k\left(F_{\alpha,\gamma}\left(I_{N,\beta,\Lambda}\left(F^{-1}_{\alpha,\gamma}v\right)-F^{-1}_{\alpha,\gamma}v\right)\right)\right\|_{\omega_{\alpha,\gamma},\Lambda}\\&=\left\|\sum_{j=0}^{k}C_k^j\partial_x^{k-j}F_{\alpha,\gamma}\partial_x^j\left(I_{N,\beta,\Lambda}\left(F^{-1}_{\alpha,\gamma}v\right)-F^{-1}_{\alpha,\gamma}v\right)\right\|_{\omega_{\alpha,\gamma},\Lambda}\\&\le c\sum_{j=0}^{k}C_k^j\left\|\partial_x^j\left(I_{N,\beta,\Lambda}\left(F^{-1}_{\alpha,\gamma}v\right)-F^{-1}_{\alpha,\gamma}v\right)\right\|_{\Lambda}\\&\le c(\beta^k+1)(\beta^2N)^{\frac13+\frac{k-r}{2}}\left\|F^{-1}_{\alpha,\gamma}v\right\|_{H^r_{A,\beta}(\Lambda)}.\end{aligned}$$

This leads to the desired result (3.20). □

Furthermore, we use (3.19), (3.18) and (3.20) successively, to derive that

$$\begin{aligned}|(v,\phi)_{\omega_{\alpha,\gamma},\Lambda}-(v,\phi)_{N,\alpha,\beta,\gamma,\Lambda}|&=|(v,\phi)_{\omega_{\alpha,\gamma},\Lambda}-(\hat I_{N,\alpha,\beta,\gamma,\Lambda}v,\phi)_{N,\alpha,\beta,\gamma,\Lambda}|\\&=|(v-\hat I_{N,\alpha,\beta,\gamma,\Lambda}v,\phi)_{\omega_{\alpha,\gamma},\Lambda}|\end{aligned}$$

$$\leq c\|v - \hat{I}_{N,\alpha,\beta,\gamma,\Lambda} v\|_{\omega_{\alpha,\gamma},\Lambda} \|\phi\|_{\omega_{\alpha,\gamma},\Lambda}$$
$$\leq c(\beta^2 N)^{\frac{1}{3}-\frac{r}{2}} \|F_{\alpha,\gamma}^{-1} v\|_{H^r_{A,\beta}(\Lambda)} \|\phi\|_{\omega_{\alpha,\gamma},\Lambda}, \quad \forall \phi \in \hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda). \tag{3.21}$$

In the end of this section, we derive an inequality which will be used in Sect. 4.

Proposition 3.1 *For any $\phi \in \hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda)$ and $1 \leq p \leq q \leq \infty$,*

$$\left\| \phi e^{\left(\frac{1}{2}-\frac{1}{q}\right)\beta^2 x^2} F_{\alpha,\gamma}^{-1+\frac{2}{q}} \right\|_{L^q_{\omega_{\alpha,\gamma}}(\Lambda)} \leq c \left(\beta N^{\frac{5}{6}}\right)^{\frac{1}{p}-\frac{1}{q}} \left\| \phi e^{\left(\frac{1}{2}-\frac{1}{p}\right)\beta^2 x^2} F_{\alpha,\gamma}^{-1+\frac{2}{p}} \right\|_{L^p_{\omega_{\alpha,\gamma}}(\Lambda)} \tag{3.22}$$

Proof Let $\mathcal{S}_N^\beta(\Lambda) = \text{span}\{\, H_l(\beta x) \mid 0 \leq l \leq N \,\}$. Due to Theorem 2.23 of [6], we have

$$\left(\int_\Lambda |\psi(x)|^q e^{-\beta^2 x^2} dx\right)^{\frac{1}{q}} \leq c(\beta N^{\frac{5}{6}})^{\frac{1}{p}-\frac{1}{q}} \left(\int_\Lambda |\psi(x)|^p e^{-\beta^2 x^2} dx\right)^{\frac{1}{p}}, \quad \forall \psi \in \mathcal{S}_N^\beta(\Lambda).$$

For any $\phi \in \hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda)$, there exists $\psi \in \mathcal{S}_N^\beta(\Lambda)$ such that $\psi(x) = e^{\frac{1}{2}\beta^2 x^2} F_{\alpha,\gamma}^{-1}(x)\phi(x)$. This leads to the desired result. □

We use (3.1) to assert that there exists positive constant $\xi_{\alpha,\gamma}$ depending only on α and γ, such that $e^{\frac{1}{4}\beta^2 x^2} F_{\alpha,\gamma}^{-\frac{1}{2}} > \dfrac{1}{\xi_{\alpha,\gamma}}$ for all $x \in \Lambda$. Consequently, we obtain from (3.22) that

$$\|\phi\|_{L^4_{\omega_{\alpha,\gamma}}(\Lambda)} \leq c\xi_{\alpha,\gamma} \left\| \phi e^{\frac{1}{4}\beta^2 x^2} F_{\alpha,\gamma}^{-\frac{1}{2}} \right\|_{L^4_{\omega_{\alpha,\gamma}}(\Lambda)} \leq c\xi_{\alpha,\gamma} \beta^{\frac{1}{4}} N^{\frac{5}{24}} \|\phi\|_{\omega_{\alpha,\gamma},\Lambda}. \tag{3.23}$$

3.4 Numerical Test

We now check the efficiency of the new approximation given by (3.7). We consider the test function

$$v(x) = \left(1 + \frac{2}{\pi}\arctan x\right)(1 + x^2)^\mu \sin kx, \tag{3.24}$$

which oscillates as $|x|$ increases. Moreover, its amplitude varies like $|x|^{2\mu}$ as $x \to +\infty$, and varies like $|x|^{2\mu-1}$ as $x \to -\infty$. Accordingly, this test function belongs to the weighted space $L^2_\chi(\Lambda)$, provided that the weight function $\chi(x)$ behaves like $|x|^\eta(\eta < -4\mu - 1)$ as $x \to +\infty$, and behaves like $|x|^\xi(\xi < -4\mu + 1)$ as $x \to -\infty$. According to (3.3), the weight function $\omega_{\alpha,\gamma}(x) \sim c|x|^{2\gamma}$ as $x \to +\infty$, and $\omega_{\alpha,\gamma}(x) \sim c|x|^{2\alpha}$ as $x \to -\infty$. Hence, the test function (3.24) could be approximated by the generalized Hermite orthogonal approximation (3.7) with $\gamma < -2\mu - \frac{1}{2}$ and $\alpha < -2\mu + \frac{1}{2}$.

We measure the errors of the orthogonal approximation by the global weighted error

$$E_{N,gw} = \left(\sum_{j=0}^{N} \left(v\left(\hat{\sigma}_{N,j}^\beta\right) - \hat{P}_{N,\alpha,\beta,\gamma} v\left(\hat{\sigma}_{N,j}^\beta\right)\right)^2 \hat{\omega}_{N,j}^{\alpha,\beta,\gamma}\right)^{\frac{1}{2}},$$

and the point-wise error

$$E_{N,pw} = \max_{0 \leq j \leq N} |v(\hat{\sigma}_{N,j}^\beta) - \hat{P}_{N,\beta,\gamma,\Lambda} v(\hat{\sigma}_{N,j}^\beta)|.$$

Table 1 Global weighted errors with $\alpha = 1$ and $\gamma = 0$

	$\beta = 1$			$\beta = \frac{3}{2}$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$N = 20$	$1.21E{-2}$	$3.03E{-2}$	$6.28E{-2}$	$8.76E{-4}$	$2.29E{-3}$	$5.24E{-3}$
$N = 60$	$3.59E{-4}$	$9.76E{-4}$	$2.27E{-3}$	$2.75E{-6}$	$7.48E{-6}$	$1.98E{-5}$
$N = 100$	$2.49E{-5}$	$6.95E{-5}$	$1.68E{-4}$	$7.78E{-8}$	$1.43E{-7}$	$3.66E{-7}$
$N = 140$	$2.64E{-6}$	$7.45E{-6}$	$1.84E{-5}$	$1.73E{-7}$	$8.97E{-8}$	$1.48E{-7}$
$N = 180$	$3.61E{-7}$	$1.03E{-6}$	$2.58E{-6}$	$1.42E{-8}$	$2.80E{-8}$	$4.09E{-8}$

Table 2 Point-wise errors with $\alpha = 1$ and $\gamma = 0$

	$\beta = 1$			$\beta = \frac{3}{2}$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$N = 20$	$9.08E{-3}$	$2.30E{-2}$	$4.81E{-2}$	$6.74E{-4}$	$1.80E{-3}$	$4.14E{-3}$
$N = 60$	$2.49E{-4}$	$6.79E{-4}$	$1.58E{-3}$	$2.16E{-6}$	$5.59E{-6}$	$1.40E{-5}$
$N = 100$	$1.69E{-5}$	$4.72E{-5}$	$1.14E{-4}$	$9.69E{-8}$	$1.72E{-7}$	$3.02E{-7}$
$N = 140$	$1.76E{-6}$	$4.98E{-6}$	$1.23E{-5}$	$2.40E{-7}$	$1.36E{-7}$	$2.00E{-7}$
$N = 180$	$2.39E{-7}$	$6.84E{-7}$	$1.71E{-6}$	$2.27E{-8}$	$4.51E{-8}$	$6.73E{-8}$

Table 3 Global weighted errors with $\alpha = \gamma = 0$

	$\beta = 1$			$\beta = \frac{3}{2}$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$N = 20$	$1.39E{-2}$	$3.45E{-2}$	$6.97E{-2}$	$1.09E{-3}$	$2.89E{-3}$	$6.47E{-3}$
$N = 60$	$2.47E{-4}$	$1.20E{-3}$	$2.75E{-3}$	$4.86E{-6}$	$1.10E{-5}$	$2.62E{-5}$
$N = 100$	$3.22E{-5}$	$8.89E{-5}$	$2.12E{-4}$	$1.02E{-7}$	$2.29E{-7}$	$4.77E{-7}$
$N = 140$	$3.48E{-6}$	$9.77E{-6}$	$2.38E{-5}$	$2.64E{-7}$	$1.33E{-7}$	$2.32E{-7}$
$N = 180$	$4.87E{-7}$	$1.38E{-6}$	$3.43E{-6}$	$2.11E{-8}$	$4.13E{-8}$	$5.99E{-8}$

In Tables 1 and 2, we list the values of $E_{N,gw}$ and $E_{N,pw}$ vs. the mode N, with the parameters $\mu = -3, \alpha = 1, \gamma = 0$ and different k. They show the convergence of the new approximation. We also find that the suitable choice of parameter β leads to better numerical results sometimes. Moreover, the approximation is better for the less oscillating solutions, as is predicted.

Remark 3.1 The text function (3.24) with $\mu < -\frac{1}{2}$ could be approximated also by the standard Hermite functions, i.e., $\alpha = \gamma = 0$. In Tables 3 and 4, we list the values of $E_{N,gw}$ and $E_{N,pw}$ vs. the mode N, with the parameters $\mu = -3, \alpha = \gamma = 0$ and different k. We see from Tables 1-4 that the numerical results with $\alpha = 1, \gamma = 0$ are better than those of the numerical results with $\alpha = \gamma = 0$.

Remark 3.2 If the asymptotic behaviors of approximated function are not fully known, then we could adopt the standard orthogonal approximation using the Hermite polynomials. How-

Table 4 Point-wise errors with $\alpha = \gamma = 0$

	$\beta = 1$			$\beta = \frac{3}{2}$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$N = 20$	1.02E−2	2.52E−2	5.08E−2	8.21E−4	2.20E−3	4.92E−3
$N = 60$	3.16E−4	8.47E−4	1.94E−3	3.17E−6	8.26E−6	1.93E−5
$N = 100$	2.30E−5	6.35E−5	1.51E−4	9.63E−8	1.81E−7	2.95E−7
$N = 140$	2.48E−6	6.97E−6	1.70E−5	2.45E−7	1.38E−7	2.06E−7
$N = 180$	3.46E−7	9.81E−7	2.43E−6	2.30E−8	4.58E−8	6.82E−8

Table 5 Global $\omega_{\alpha,\gamma}(x)$-weighted errors of approximation (3.25)

	$N = 20$	$N = 60$	$N = 100$	$N = 140$	$N = 180$
$k = 1$	1.74E−2	6.35E−4	4.81E−5	5.36E−6	7.65E−7
$k = 2$	4.17E−2	1.68E−3	1.32E−4	1.49E−5	2.16E−6
$k = 3$	8.08E−2	3.77E−3	3.10E−4	3.61E−5	5.30E−6

Table 6 Point-wise errors of approximation (3.25)

	$N = 20$	$N = 60$	$N = 100$	$N = 140$	$N = 180$
$k = 1$	1.27E−2	4.48E−4	3.43E−5	3.81E−6	5.42E−7
$k = 2$	3.05E−2	1.18E−3	9.39E−5	1.06E−5	1.53E−6
$k = 3$	5.89E−2	2.65E−3	2.21E−4	2.56E−5	3.75E−6

ever, in this case, the small global e^{-x^2}-weighted errors do not imply the small point-wise errors automatically. But, we may follow the idea of London to approximate the auxiliary function $v^*(x) = v(x)\text{sech}x$ by using the Hermite functions, see [22]. More precisely, we let $\hat{v}_l^*$ be the coefficients of the expansion of $v^*(x)$ in terms of $e^{-\frac{1}{2}x^2} H_l(x)$. Then we obtain the following approximation to the original function,

$$P_N^* v(x) = \frac{1}{\text{sech}x} \left(\sum_{l=0}^{N} \hat{v}_l^* e^{-\frac{1}{2}x^2} H_l(x) \right). \tag{3.25}$$

We could use (2.5) with $k = 0$ and $\beta = 1$ to verify that

$$||P_N^* v - v||^2_{\text{sech}x,\,\Lambda} = \| P_{N,1,\Lambda} v^* - v^* \|_\Lambda \le c N^{-\frac{r}{2}} \| v \text{sech} x \|_{H^r_{A,1}(\Lambda)}.$$

Since the weight function $\text{sech}^2 x$ decays like $e^{-2|x|}$ as $|x| \to \infty$, the above global weighted errors are small usually. But the corresponding global errors with the weight function $\omega_{\alpha,\gamma}(x)$ are bigger than those of the approximation (3.7). In Table 5, we list the global $\omega_{\alpha,\gamma}(x)$-weighted errors of the approximation (3.25) with the mode $N = 10$ and $\beta = 1$ for the test function with $\mu = -3$. They are really bigger than those of the approximation (3.7) as listed in Table 1.

In Table 6, we list the corresponding point-wise errors of the approximation (3.25). We find that the point-wise errors of the approximation (3.25) are also bigger than those of the approximation (3.7). Furthermore, in the applications of the above two approximations to numerical solutions of differential equations defined on the whole line, we have to multiply the underlying differential equations by the weight functions and integrate the resulting equalities by parts, and then derive their weak formulations. Moreover, in the numerical analysis of the corresponding spectral methods, we need some results on the $H^1_{\text{sech}^2 x}(\Lambda)$-orthogonal approximation and the $H^1_{\omega_{\alpha,\gamma}}(\Lambda)$-orthogonal approximation, respectively. For this purpose, it seems simpler to use the approximation with the weight function $\omega_{\alpha,\gamma}(x)$ usually.

Remark 3.3 We may also use the generalized Jacobi irrational approximation proposed in [12]. For $a, b > -1$, $J_l^{(a,b)}(x)$ stands for the Jacobi polynomial of degree l. For any real numbers a and b,

$$\hat{a} := \begin{cases} -a, & a \le -1, \\ 0, & a > -1, \end{cases} \quad \bar{a} := \begin{cases} -a, & a \le -1, \\ a, & a > -1, \end{cases}$$

(likewise for $\hat{b}$ and $\bar{b}$). The symbol $[a]$ represents the largest integer $\le a$. The generalized Jacobi functions are given by

$$\tilde{J}_l^{(a,b)}(x) := \begin{cases} J_l^{(a,b)}(x), & a, b > -1, \\ (1+x)^{-b} J_{l-[-b]}^{(a,-b)}(x), & a > -1, b \le -1, \\ (1-x)^{-a} J_{l-[-a]}^{(-a,b)}(x), & a \le -1, b > -1, \\ (1-x)^{-a}(1+x)^{-b} J_{l-[-a]-[-b]}^{(-a,-b)}(x), & a, b \le -1. \end{cases}$$

The generalized Jacobi irrational functions are defined by

$$R_l^{(a,b)}(x) = \tilde{J}_l^{(a,b)}\left(\frac{x}{\sqrt{x^2+1}}\right), \quad l \ge [\hat{a}] + [\hat{b}].$$

The corresponding weight function is

$$\omega_R^{(a,b)}(x) = \left(\sqrt{x^2+1} + x\right)^{b-a} (x^2+1)^{-\frac{a+b+3}{2}}.$$

According to (3.6) of [12], we know that

$$\begin{aligned} \omega_R^{(a,b)}(x) &\sim c|x|^{-2a-3}, \quad \text{as } x \to +\infty, \\ \omega_R^{(a,b)}(x) &\sim c|x|^{-2b-3}, \quad \text{as } x \to -\infty. \end{aligned} \tag{3.26}$$

The generalized Jacobi irrational functions form a complete $L^2_{\omega_R^{(a,b)}}(\Lambda)$-orthogonal system.

We now approximate the test function (3.24) in a specific way, namely,

$$P_N^{**} v(x) = \sum_{l=0}^{N} \hat{v}_l^{**} R_l^{\left(2\mu - \frac{11}{12}, 2\mu - \frac{23}{12}\right)}(x), \tag{3.27}$$

v_l^{**} being the coefficients of the expansion of $v(x)$ in terms of $R_l^{(2\mu-\frac{11}{12}, 2\mu-\frac{23}{12})}(x)$. According to (3.3) and (3.26), the asymptotic behaviors of the weight function $\omega_R^{(2\mu-\frac{11}{12}, 2\mu-\frac{23}{12})}(x)$ used in the approximation (3.27) are exactly the same as the asymptotic behaviors of the weight function $\omega_{-2\mu+\frac{5}{12}, -2\mu-\frac{7}{12}}(x)$ used in the approximation (3.7). Thus, we could compare the accuracy of the above two approximations.

Let $\mu = -3$ in the test function (3.24). In Tables 7 and 8, we list the global $\omega_R^{(-\frac{83}{12}, -\frac{95}{12})}(x)$-weight errors and the point-wise errors of the approximation (3.27). By comparing Tables 1 and 2 with Tables 7 and 8, we find that the approximations (3.27) provides the better numerical results than those of the approximations (3.7) with $\beta = 1$. But the approximations (3.7) with $\beta = \frac{3}{2}$ provides the better numerical results than those of the approximations (3.27). However,

Table 7 Global $\omega_R^{\left(-\frac{83}{12},-\frac{95}{12}\right)}(x)$-weighted errors of approximation (3.27)

	$N=20$	$N=60$	$N=100$	$N=140$	$N=180$
$k=1$	$1.30E{-3}$	$4.64E{-4}$	$3.14E{-6}$	$3.28E{-7}$	$8.91E{-8}$
$k=2$	$2.11E{-3}$	$2.82E{-4}$	$7.38E{-5}$	$7.10E{-6}$	$3.24E{-7}$
$k=3$	$4.96E{-3}$	$3.45E{-4}$	$4.34E{-5}$	$3.11E{-6}$	$6.76E{-7}$

Table 8 Point-wise errors of approximation (3.27)

	$N=20$	$N=60$	$N=100$	$N=140$	$N=180$
$k=1$	$1.23E{-3}$	$4.46E{-4}$	$2.41E{-6}$	$2.01E{-7}$	$4.42E{-8}$
$k=2$	$2.05E{-3}$	$2.18E{-4}$	$5.39E{-5}$	$1.12E{-6}$	$1.34E{-7}$
$k=3$	$4.82E{-3}$	$2.74E{-4}$	$2.12E{-5}$	$1.56E{-6}$	$5.75E{-7}$

in their applications to numerical solutions of differential equations, we have to derive the weak formulations of underlying problems, and need some results on the $H^1_{\omega_R^{(-\frac{83}{12},-\frac{95}{12})}}(\Lambda)$-orthogonal approximation and the $H^1_{\omega_{-\frac{67}{12},-\frac{79}{12}}}(\Lambda)$-orthogonal approximation, respectively. It seems simpler to use the approximation (3.7) than the approximation (3.27). Besides, in actual computation, it is easier to perform the generalized Hermite orthogonal approximation than the generalized Jacobi irrational orthogonal approximation.

4 Generalized Hermite Spectral Method

In this section, we propose the new generalized Hermite spectral method.

4.1 A Linear Problem

Let d be a positive constant. We consider the following model problem,

$$\begin{cases} -\partial_x^2 U(x) + dU(x) = f(x), & x \in \Lambda, \\ U(x)x^{-\mu} \to 0, & \text{as } x \to +\infty, \\ U(x)x^{-\nu} \to 0, & \text{as } x \to -\infty, \end{cases} \tag{4.1}$$

where $f \in L^2_{\omega_{\alpha,\gamma}}(\Lambda)$, $\nu < -\alpha - \dfrac{1}{2}$ and $\mu < -\gamma - \dfrac{1}{2}$.

In order to derive a proper weak formulation of (4.1), we set $V(\Lambda) = H^1_{\omega_{\alpha,\gamma}}(\Lambda)$, and introduce the bilinear form

$$\begin{aligned} \mathcal{A}_{d,\alpha,\gamma,\Lambda}(u,v) = &\int_\Lambda \partial_x u(x)\partial_x v(x)\omega_{\alpha,\gamma}(x)dx + \int_\Lambda \partial_x u(x)v(x)\partial_x\omega_{\alpha,\gamma}(x)dx \\ &+ d\int_\Lambda u(x)v(x)\omega_{\alpha,\gamma}(x)dx, \quad \forall u,v \in V(\Lambda). \end{aligned} \tag{4.2}$$

Let

$$G_{\alpha,\gamma}(x) = \frac{4}{\pi(1+x^2)}\left(\gamma\left(1-\frac{2}{\pi}\arctan x\right)^{-1} - \alpha\left(1+\frac{2}{\pi}\arctan x\right)^{-1}\right).$$

Then

$$\partial_x\omega_{\alpha,\gamma}(x) = \omega_{\alpha,\gamma}(x)G_{\alpha,\gamma}(x), \quad \partial_x^2\omega_{\alpha,\gamma}(x) = \omega_{\alpha,\gamma}(x)\left(\partial_x G_{\alpha,\gamma}(x) + G^2_{\alpha,\gamma}(x)\right). \tag{4.3}$$

Thanks to (3.1), we observe that

$$
\begin{aligned}
G_{\alpha,\gamma}(x) &\sim \frac{2}{\pi(1+x^2)}(\pi\gamma(1+x)-\alpha), \quad \text{as } x \to +\infty,\\
G_{\alpha,\gamma}(x) &\sim \frac{2}{\pi(1+x^2)}(\gamma-\pi\alpha(1+x)), \quad \text{as } x \to -\infty.
\end{aligned}
$$

Hence, there exists a positive constant $c_{\alpha,\gamma}$ depending only on α and γ, such that

$$
|G_{\alpha,\gamma}(x)| \le c_{\alpha,\gamma}, \quad \forall x \in \Lambda. \tag{4.4}
$$

Furthermore,

$$
\begin{aligned}
\partial_x G_{\alpha,\gamma}(x) = \frac{-2x}{1+x^2}G_{\alpha,\gamma}(x) + \frac{8}{\pi^2(1+x^2)^2}\Bigg(\gamma\left(1-\frac{2}{\pi}\arctan x\right)^{-2}\\
+\alpha\left(1+\frac{2}{\pi}\arctan x\right)^{-2}\Bigg).
\end{aligned}
$$

Due to (3.1), we have that

$$
\begin{aligned}
\partial_x G_{\alpha,\gamma}(x) \sim &-\frac{4x}{\pi(1+x^2)^2}(\pi\gamma(1+x)-\alpha)\\
&+\frac{2}{\pi^2(1+x^2)^2}(\gamma\pi^2(1+x)^2+\alpha), \quad \text{as } x \to +\infty,\\
\partial_x G_{\alpha,\gamma}(x) \sim &-\frac{4x}{\pi(1+x^2)^2}(\gamma-\pi\alpha(1+x))\\
&+\frac{2}{\pi^2(1+x^2)^2}(\gamma+\alpha\pi^2(1+x)^2), \quad \text{as } x \to -\infty.
\end{aligned}
$$

Thus, there exists a positive constant $d_{\alpha,\gamma}$ depending only on α and γ, such that

$$
|\partial_x G_{\alpha,\gamma}(x) + G^2_{\alpha,\gamma}(x)| \le 2d_{\alpha,\gamma}, \quad \forall x \in \Lambda. \tag{4.5}
$$

With the aid of (4.4) and the Cauchy inequality, a direct calculation shows

$$
\begin{aligned}
|\mathcal{A}_{d,\alpha,\gamma,\Lambda}(u,v)| \le \frac{1}{2}\Big((1+c_{\alpha,\gamma})\|\partial_x u\|^2_{\omega_{\alpha,\gamma},\Lambda} + d\|u\|^2_{\omega_{\alpha,\gamma},\Lambda} + \|\partial_x v\|^2_{\omega_{\alpha,\gamma},\Lambda}\\
+(d+c_{\alpha,\gamma})\|v\|^2_{\omega_{\alpha,\gamma},\Lambda}\Big), \quad \forall u, v \in V(\Lambda).
\end{aligned} \tag{4.6}
$$

Next, we use (4.5) to derive that for any $v \in V(\Lambda)$,

$$
\begin{aligned}
\int_\Lambda \partial_x v(x)v(x)\partial_x\omega_{\alpha,\gamma}(x)dx &= -\frac{1}{2}\int_\Lambda v^2(x)\partial_x^2\omega_{\alpha,\gamma}(x)dx\\
&= -\frac{1}{2}\int_\Lambda v^2(x)\omega_{\alpha,\gamma}(x)\left(\partial_x G_{\alpha,\gamma}(x)+G^2_{\alpha,\gamma}(x)\right)dx\\
&\ge -d_{\alpha,\gamma}\|v\|^2_{\omega_{\alpha,\gamma},\Lambda}.
\end{aligned} \tag{4.7}
$$

Inserting (4.7) into (4.2) with $u = v$, we obtain

$$
\mathcal{A}_{d,\alpha,\gamma,\Lambda}(v,v) \ge \|\partial_x v\|^2_{\omega_{\alpha,\gamma},\Lambda} + (d-d_{\alpha,\gamma})\|v\|^2_{\omega_{\alpha,\gamma},\Lambda}, \quad \forall v \in V(\Lambda). \tag{4.8}
$$

We now derive another property of the bilinear form $\mathcal{A}_{d,\alpha,\gamma,\Lambda}(u,v)$, which plays an important role in spectral method. Let $W(\Lambda) \subseteq V(\Lambda)$, and $\mathcal{Q}^*_N(\Lambda)$ be a finite-dimensional subspace of $V(\Lambda)$. In addition, $W_N(\Lambda) = W(\Lambda) \cap \mathcal{Q}_N^*(\Lambda)$. We define the operator ${}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda} : W(\Lambda) \to W_N(\Lambda)$, by

$$\mathcal{A}_{d,\gamma,\Lambda}\left({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v, \phi\right) = 0, \quad \forall\, \phi \in W_N(\Lambda). \tag{4.9}$$

Proposition 4.1 *If $v \in W(\Lambda)$, $w \in W_N(\Lambda)$ and $d > d_{\alpha,\gamma}$, then*

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}\left({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v, *P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v\right) \leq \mathcal{A}_{d,\alpha,\gamma,\Lambda}(w - v, w - v). \tag{4.10}$$

Proof A direct calculation shows that

$$\begin{aligned}\mathcal{A}_{d,\alpha,\gamma,\Lambda}(w - v, w - v) = \mathcal{A}_{d,\alpha,\gamma,\Lambda}&\left({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v, {}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v\right)\\ &+ \mathcal{A}_{d,\alpha,\gamma,\Lambda}\left({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - w, {}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - w\right)\\ &- 2\mathcal{A}_{d,\alpha,\gamma,\Lambda}\left({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v, {}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - w\right).\end{aligned}$$

Thanks to (4.9), we have

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v, {}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - w) = 0.$$

Due to $d > d_{\alpha,\gamma}$, the property (4.8) implies

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}({}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - w, {}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - w) \geq 0.$$

Then, the desired inequality (4.10) follows from the previous statements immediately. □

Evidently, for any $u, v \in V(\Lambda)$, we have $\partial_x u(x)v(x)\omega_{\alpha,\gamma}(x) \to 0$ as $|x| \to \infty$. By multiplying the equation in (4.1) by $v(x)\omega_{\alpha,\gamma}(x)$ and integrating the resulting equality by parts, we obtain a weak formulation of (4.1). It is to look for $U \in V(\Lambda)$ such that

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}(U, v) = (f, v)_{\omega_{\alpha,\gamma},\Lambda}, \quad \forall v \in V(\Lambda). \tag{4.11}$$

According to (4.6), (4.8) and the Lax-Milgram lemma, the problem (4.11) admits a unique solution as long as $d > d_{\alpha,\gamma}$.

For solving the above problem with $d > d_{\alpha,\gamma}$ numerically, we define the finite-dimensional set

$$V_N(\Lambda) = V(\Lambda) \cap \hat{\mathcal{Q}}_N^{\alpha,\beta,\gamma}(\Lambda).$$

The spectral method for solving problem (4.11) is to seek $u_N \in V_N(\Lambda)$ such that

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}(u_N, \phi) = (f, \phi)_{\omega_{\alpha,\gamma},\Lambda}, \quad \forall \phi \in V_N(\Lambda). \tag{4.12}$$

For checking the existence of solutions of (4.12), it suffices to prove the uniqueness of its solutions. Assume that both of $u_N^{(1)}(x)$ and $u_N^{(2)}(x)$ are solutions of (4.12), and $\tilde{u}_N(x) = u_N^{(1)}(x) - u_N^{(2)}(x) \in V_N(\Lambda)$. Then $\mathcal{A}_{d,\alpha,\gamma,\Lambda}(\tilde{u}_N, \phi) = 0$ for any $\phi \in V_N(\Lambda)$. Putting $\phi = \tilde{u}_N \in V_N(\Lambda)$ in the above equation, we use (4.8) to obtain

$$\|\partial_x \tilde{u}_N\|^2_{\omega_{\alpha,\gamma},\Lambda} + (d - d_{\alpha,\gamma})\|\tilde{u}_N\|^2_{\omega_{\alpha,\gamma},\Lambda} \leq \mathcal{A}_{d,\alpha,\gamma,\Lambda}(\tilde{u}_N, \tilde{u}_N) = 0.$$

Since $d > d_{\alpha,\gamma}$, we have $\tilde{u}_N(x) \equiv 0$. This means the uniqueness of solution of (4.12).

We next estimate the error of numerical solution $u_N(x)$. To do this, we introduce the auxiliary operator $\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda} : V(\Lambda) \to V_N(\Lambda)$, defined by

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}\left(\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}v - v, \phi\right) = 0, \quad \forall\, \phi \in V_N(\Lambda). \tag{4.13}$$

We have from (4.11) and (4.13) that

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}\left(\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U,\phi\right)=(f,\phi)_{\omega_{\alpha,\gamma},\Lambda},\quad \forall\phi\in V_N(\Lambda). \tag{4.14}$$

Subtracting (4.14) from (4.12), yields

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}\left(u_N-\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U,\phi\right)=0,\quad \forall\phi\in V_N(\Lambda).$$

Taking $\phi=u_N-\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U$ in the above equation, we obtain

$$\mathcal{A}_{d,\alpha,\gamma,\Lambda}\left(u_N-\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U,u_N-\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U\right)=0.$$

This fact, together with (4.8), implies $u_N=\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U$.

So far, it remains to estimate the approximation error of the auxiliary operator $\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U$. For this purpose, we use Proposition 4.1 with

$$\begin{aligned}&W(\Lambda)=V(\Lambda),\quad \mathcal{Q}^*_N(\Lambda)=\hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_N(\Lambda),\quad W_N(\Lambda)=V_N(\Lambda),\\ &v=U,\quad {}_*P^1_{N,d,\alpha,\beta,\gamma,\Lambda}v=\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U,\quad w=\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}U.\end{aligned}$$

Then, by using (4.8), (4.10) and (4.6) successively, we verify that

$$\begin{aligned}&\left\|\partial_x\left(\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right)\right\|^2_{\omega_{\alpha,\gamma},\Lambda}+(d-d_{\alpha,\gamma})\left\|\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right\|^2_{\omega_{\alpha,\gamma},\Lambda}\\ &\le \mathcal{A}_{d,\alpha,\gamma,\Lambda}\left(\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U,\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right)\\ &\le \mathcal{A}_{d,\alpha,\gamma,\Lambda}\left(\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}U-U,\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}U-U\right)\\ &\le \left(1+\frac{1}{2}c_{\alpha,\gamma}\right)\left\|\partial_x\left(\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}U-U\right)\right\|^2_{\omega_{\alpha,\gamma},\Lambda}\\ &\quad+\left(d+\frac{1}{2}c_{\alpha,\gamma}\right)\left\|\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}U-U\right\|^2_{\omega_{\alpha,\gamma},\Lambda}.\end{aligned} \tag{4.15}$$

Finally, we use (4.15) and (3.12) to reach that for $d>d_{\alpha,\gamma}$ and integer $r\ge 2$,

$$\begin{aligned}\|U-u_N\|^2_{H^1_{\omega_{\alpha,\gamma}}(\Lambda)}&=\left\|\partial_x\left(\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right)\right\|^2_{\omega_{\alpha,\gamma},\Lambda}+\left\|\overline{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right\|^2_{\omega_{\alpha,\gamma},\Lambda}\\ &\le\left(1+\frac{1}{d-d_{\alpha,\gamma}}\right)\left(\left(1+\frac{1}{2}c_{\alpha,\gamma}\right)\left\|\partial_x\left(\hat{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right)\right\|^2_{\omega_{\alpha,\gamma},\Lambda}\right.\\ &\qquad\left.+\left(d+\frac{1}{2}c_{\alpha,\gamma}\right)\left\|\hat{P}^1_{N,d,\alpha,\beta,\gamma,\Lambda}U-U\right\|^2_{\omega_{\alpha,\gamma},\Lambda}\right)\\ &\le c\left(1+\frac{1}{d-d_{\alpha,\gamma}}\right)\left(d+1+\frac{1}{2}c_{\alpha,\gamma}\right)(\beta^2N)^{\frac{1-r}{2}}\left\|F^{-1}_{\alpha,\gamma}U\right\|_{H^r_{A,\beta}(\Lambda)}.\end{aligned} \tag{4.16}$$

Remark 4.1 In the numerical analysis of spectral scheme (4.12), we require $d>d_{\alpha,\gamma}$. But this restriction is not essential. Indeed, we may take $\theta>\sqrt{\frac{d_{\alpha,\gamma}}{d}}$ and $\tilde{d}=\theta^2 d$. Let $y=\dfrac{x}{\theta}$, $V(y)=U(x)$ and $F(y)=\theta^2 f(x)$. Then the problem (4.1) becomes

$$\begin{cases} -\partial_y^2 V(y) + \tilde{d} V(y) = F(y), & y \in \Lambda, \\ V(y) y^{-\mu} \to 0, & \text{as } y \to +\infty, \\ V(y) y^{-\nu} \to 0, & \text{as } y \to -\infty. \end{cases}$$

Clearly, $\tilde{d} > d_{\alpha,\gamma}$. Thus, the previous analysis is also valid for the corresponding spectral scheme of the above reformed problem.

We now describe the implementation for the spectral scheme (4.12). Let

$$\phi_l(x) = \pi^{-\frac{1}{4}} \hat{H}_l^{\alpha,\beta,\gamma}(x), \quad 0 \le l \le N.$$

We expand the numerical solution as

$$u_N(x) = \sum_{l=0}^{N} \hat{u}_l \phi_l(x).$$

Inserting the above expression into (4.12) with $\phi = \phi_k(x)$, we obtain

$$\sum_{l=0}^{N} \Big((\partial_x \phi_l, \partial_x \phi_k)_{\omega_{\alpha,\gamma},\Lambda} + (\partial_x \phi_l, \phi_k G_{\alpha,\gamma})_{\omega_{\alpha,\gamma},\Lambda} + d(\phi_l, \phi_k)_{\omega_{\alpha,\gamma},\Lambda} \Big) \hat{u}_l \\ = (f, \phi_k)_{\omega_{\alpha,\gamma},\Lambda}, 0 \le k \le N. \tag{4.17}$$

We can rewrite the system (4.17) as a compact matrix form. To do this, we introduce the matrices $A = (a_{k,l})_{0 \le k,l \le N}$, $B = (b_{k,l})_{0 \le k,l \le N}$ and $C = (c_{k,l})_{0 \le k,l \le N}$, with the following entries:

$$a_{k,l} = (\partial_x \phi_l, \partial_x \phi_k)_{\omega_{\alpha,\gamma},\Lambda}, \quad b_{k,l} = (\partial_x \phi_l, \phi_k G_{\alpha,\gamma})_{\omega_{\alpha,\gamma},\Lambda}, \quad c_{k,l} = (\phi_l, \phi_k)_{\omega_{\alpha,\gamma},\Lambda}.$$

Further, let $\mathbf{\hat{u}} = (\hat{u}_0, \hat{u}_1, \cdots, \hat{u}_N)^T$ and $\mathbf{F} = (F_0, F_1, \cdots, F_N)^T$ with $F_k = (f, \phi_k)_{\omega_{\alpha,\gamma},\Lambda}$. Then, the system (4.17) becomes

$$(A + B + dC)\mathbf{\hat{u}} = \mathbf{F}.$$

We now use the spectral scheme (4.12) (equivalent to the system (4.17)) to solve problem (4.11). We take the test function

$$U(x) = \left(1 + \frac{2}{\pi} \arctan x\right) (1 + x^2)^{-3} \sin x. \tag{4.18}$$

According to (3.1) and (3.3), we know that $U \in L^2_{\omega_{\alpha,\gamma}}(\Lambda)$ for any $\alpha < \frac{13}{2}$ and $\gamma < \frac{11}{2}$. Thus, we could use the spectral scheme (4.12) with such parameters α and γ.

We use the spectral scheme (4.12) with $\alpha = 4$ and $\gamma = 3$ to solve problem (4.11) with $d = 1$. For comparison, we also solve the same problem (4.11) by the spectral scheme (4.12) with $\alpha = \gamma = 0$, which is equivalent to the spectral method using the Hermite functions proposed in [9]. We shall measure the numerical accuracy by the same global weighted errors

$$E_{N,gw} = \left(\sum_{j=0}^{N} \left(U\left(\hat{\sigma}_{N,j}^{\beta}\right) - u_N\left(\hat{\sigma}_{N,j}^{\beta}\right) \right)^2 \hat{\omega}_{N,j}^{\alpha,\beta,\gamma} \right)^{\frac{1}{2}},$$

and the same point-wise errors

$$E_{N,pw} = \max_{0 \le j \le N} \left| U\left(\hat{\sigma}_{N,j}^{\beta}\right) - u_N\left(\hat{\sigma}_{N,j}^{\beta}\right) \right|.$$

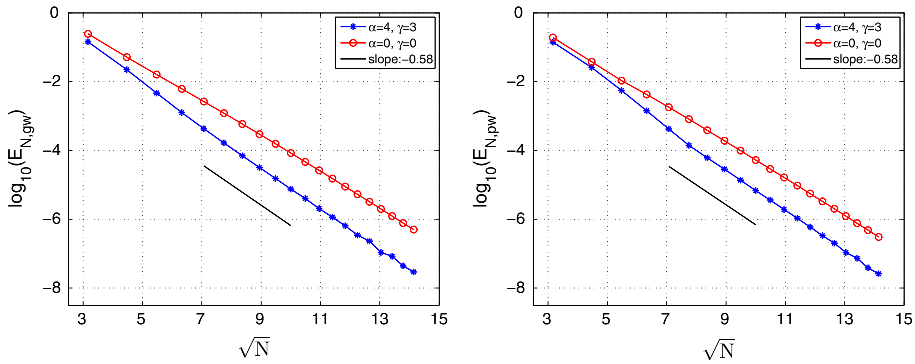


Fig. 1 Comparison of numerical errors. *Left* global weighted errors; *Right* point-wise errors

In Fig. 1, we plot the values of $E_{N,gw}$ and $E_{N,pw}$ with $\beta = 1$ vs $\sqrt{N}$. The numerical results demonstrate the convergence of the scheme (4.12) with the above parameters α and γ. They also show that the numerical results with $\alpha = 4$ and $\gamma = 3$ are better than the numerical results with $\alpha = \gamma = 0$. In fact, the base functions $\hat{H}_l^{4,1,3}(x)$ simulate the asymptotic behavior of the test function (4.18) more reasonably than the base functions $\hat{H}_l^{0,1,0}(x)$. This is one of the advantages of our new method (4.12).

4.2 Fisher Equation

In some practical problems, we know the boundary conditions at infinity exactly. In this case, we may use certain variable transformation to derive an alternative form of the original problems with homogeneous boundary conditions, and use the generalized Hermite approximation with $\alpha = \gamma = 0$. As an example of nonlinear problems, we consider the Fisher equation as follows,

$$\begin{cases} \partial_t U(x,t) - \partial_x^2 U(x,t) - aU(x,t)(1-U(x,t)) = f(x,t), & x \in \Lambda,\ 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \Lambda, \end{cases} \tag{4.19}$$

where the positive constant a is a measure of intensity of insects. Guo and Chen [5] found the following heteroclinic solutions of the Fisher equation with $f(x,t) \equiv 0$,

$$U(x,t) = \frac{1-\varepsilon}{2} + \frac{\varepsilon}{1 + \frac{A}{B}e^{b\eta}} - \frac{AB}{\left(Ae^{\frac{b\eta}{2}} + Be^{-\frac{b\eta}{2}}\right)^2}, \tag{4.20}$$

where A and B are arbitrary constants, $\varepsilon = \pm 1$, $b = \sqrt{\frac{a}{6}}$ and $\eta = x - \frac{5}{6}\sqrt{6a}\varepsilon t$. If $\varepsilon = 1$, then $U(x,t) \to 1$ as $x \to -\infty$, and $U(x,t) \to 0$ as $x \to \infty$. If $\varepsilon = -1$, then $U(x,t) \to 0$ as $x \to -\infty$, and $U(x,t) \to 1$ as $x \to \infty$.

We now derive the weak formulation of (4.19), which depends on the asymptotic behavior of $U(x,t)$ at infinities. We assume that

$$U(x,t)\partial_x U(x,t)\omega_{\alpha,\gamma}(x) \to 0, \qquad \text{as } |x| \to \infty,\ 0 \le t \le T,\ a.e.. \tag{4.21}$$

Indeed, if $U(x,t) \in H^1_{\omega_{\alpha,\gamma}}(\Lambda)$ for $0 \le t \le T$, then

$$U(x,t)\omega^{\frac{1}{2}}_{\alpha,\gamma}(x) = o\left(x^{-\frac{1}{2}}\right), \quad \partial_x U(x,t)\omega^{\frac{1}{2}}_{\alpha,\gamma}(x) = o\left(x^{-\frac{1}{2}}\right), \quad \text{as } |x| \to \infty,\ 0 \le t \le T.$$

In this case, the boundary condition (4.21) is satisfied.

Let $V(\Lambda) = H^1_{\omega_{\alpha,\gamma}}(\Lambda)$ as in the last subsection, and $v \in V(\Lambda)$. By multiplying the first equation of (4.19) by $v(x)\omega_{\alpha,\gamma}(x)$ and integrating the resulting equality by parts over the interval Λ, we obtain the weak formulation of problem (4.19) with the boundary condition (4.21). It is to find the solution $U \in L^\infty(0,T; L^2_{\omega_{\alpha,\gamma}}(\Lambda)) \cap L^2(0,T; V(\Lambda))$ such that

$$\begin{cases} (\partial_t U(t) - aU(t) + aU^2(t), v)_{\omega_{\alpha,\gamma},\Lambda} + (\partial_x U(t), & \partial_x(v\omega_{\alpha,\gamma}))_\Lambda \\ \quad = (f(t), v)_{\omega_{\alpha,\gamma},\Lambda}, & \forall\, v \in V(\Lambda),\ 0 < t \le T, \\ U(x,0) = U_0(x), & x \in \Lambda. \end{cases} \tag{4.22}$$

If $U_0 \in L^2_{\omega_{\alpha,\gamma}}(\Lambda)$ and $f \in L^2(0,T; L^2_{\omega_{\alpha,\gamma}}(\Lambda))$, then problem (4.22) admits a unique solution.

Let $V_N(\Lambda) = V(\Lambda) \cap \hat{\mathcal{Q}}^{\alpha,\beta,\gamma}_N(\Lambda)$ as before. The spectral scheme for solving problem (4.22) is to seek the numerical solution $u_N(t) \in V_N(\Lambda)$ for all $0 \le t \le T$, such that

$$\begin{cases} \left(\partial_t u_N(t) - au_N(t) + au_N^2(t), \phi\right)_{\omega_{\alpha,\gamma},\Lambda} + (\partial_x u_N(t), & \partial_x(\phi\omega_{\alpha,\gamma}))_\Lambda \\ \quad = (f(t), \phi)_{\omega_{\alpha,\gamma},\Lambda}, & \forall\, \phi \in V_N(\Lambda),\ 0 < t \le T, \\ u_N(x,0) = \hat{P}_{N,\alpha,\beta,\gamma,\Lambda} U_0(x) \text{ or } \hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda} U_0(x), & x \in \Lambda. \end{cases} \tag{4.23}$$

We now deal with the convergence of scheme (4.23). Let $U_N = \hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda} U$. By virtue of (3.11) and (4.3), we have from (4.22) that

$$\left(\partial_t U_N(t) - aU_N(t) + aU_N^2(t), \phi\right)_{\omega_{\alpha,\gamma},\Lambda} + (\partial_x U_N(t), \partial_x \phi)_{\omega_{\alpha,\gamma},\Lambda} + (\partial_x U_N(t), \phi G_{\alpha,\gamma})_{\omega_{\alpha,\gamma},\Lambda}$$
$$+ \sum_{j=1}^4 G_j(t,\phi) = (f(t), \phi)_{\omega_{\alpha,\gamma},\Lambda}, \quad \forall\, \phi \in V_N(\Lambda),\ 0 < t \le T, \tag{4.24}$$

where

$$\begin{aligned} G_1(t,\phi) &= (\partial_t U(t) - \partial_t U_N(t), \phi)_{\omega_{\alpha,\gamma},\Lambda}, \\ G_2(t,\phi) &= (a+1)(U_N(t) - U(t), \phi)_{\omega_{\alpha,\gamma},\Lambda}, \\ G_3(t,\phi) &= (\partial_x U(t) - \partial_x U_N(t), \phi G_{\alpha,\gamma})_{\omega_{\alpha,\gamma},\Lambda}, \\ G_4(t,\phi) &= a(U^2(t) - U_N^2(t), \phi)_{\omega_{\alpha,\gamma},\Lambda}. \end{aligned}$$

Further, we set $\tilde{U}_N = u_N - U_N$. By subtracting (4.24) from (4.23), we obtain

$$\begin{cases} (\partial_t \tilde{U}_N(t) - a\tilde{U}_N(t), \phi)_{\omega_{\alpha,\gamma},\Lambda} + (\partial_x \tilde{U}_N(t), \partial_x \phi)_{\omega_{\alpha,\gamma},\Lambda} \\ \quad = \sum_{j=1}^2 F_j(t,\phi) + \sum_{j=1}^4 G_j(t,\phi), \quad \forall \phi \in V_N(\Lambda),\ t \in (0,T], \\ \tilde{U}_N(x,0) = \hat{P}_{N,\alpha,\beta,\gamma,\Lambda} U_0(x) - \hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda} U_0(x) \text{ or } 0, \quad x \in \bar{\Lambda}, \end{cases} \tag{4.25}$$

with

$$\begin{aligned} F_1(t,\phi) &= -a(2U_N(t)\tilde{U}_N(t) + \tilde{U}_N^2(t), \phi)_{\omega_{\alpha,\gamma},\Lambda}, \\ F_2(t,\phi) &= -(\partial_x \tilde{U}_N(t), \phi G_{\alpha,\gamma})_{\omega_{\alpha,\gamma},\Lambda}. \end{aligned}$$

Taking $\phi = 2\tilde{U}_N(t)$ in (4.25), we obtain

$$\partial_t \|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + 2\|\partial_x \tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} - 2a\|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda}$$

$$= 2\sum_{j=1}^{2} F_j(t, \tilde{U}_N(t)) + 2\sum_{j=1}^{4} G_j(t, \tilde{U}_N(t)). \tag{4.26}$$

We are going to estimate the right side of the above equality. Firstly, we use the Cauchy inequality, (3.23) and (3.12) to derive that

$$\begin{aligned}
&2|F_1(t, \tilde{U}_N(t))| \\
&\quad = 2a|2(U(t), \tilde{U}_N^2(t))_{\omega_{\alpha,\gamma},\Lambda} + 2(U_N(t) - U(t), \tilde{U}_N^2(t))_{\omega_{\alpha,\gamma},\Lambda} + (\tilde{U}_N(t), \tilde{U}_N^2(t))_{\omega_{\alpha,\gamma},\Lambda}| \\
&\quad \le 4a\|U(t)\|_{L^\infty(\Lambda)}\|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + 4||U_N(t) - U(t)||_{\omega_{\alpha,\gamma},\Lambda}||\tilde{U}_N(t)||^2_{L^4_{\omega_{\alpha,\gamma}}(\Lambda)} \\
&\qquad + 4||\tilde{U}_N(t)||_{\omega_{\alpha,\gamma},\Lambda}||\tilde{U}_N(t)||^2_{L^4_{\omega_{\alpha,\gamma}}(\Lambda)} \\
&\quad \le 4a\|U(t)\|_{L^\infty(\Lambda)}\|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + ca\xi^2_{\alpha,\gamma}\beta^{\frac{3}{2}}N^{-\frac{1}{12}}\|F^{-1}_{\alpha,\gamma}U(t)\|_{H^2_{A,\beta}(\Lambda)}\|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} \\
&\qquad + ca\xi^2_{\alpha,\gamma}\beta^{\frac{1}{2}}N^{\frac{5}{12}}\|\tilde{U}_N(t)\|^3_{\omega_{\alpha,\gamma},\Lambda}.
\end{aligned} \tag{4.27}$$

Next, by virtue of (4.4), we have

$$2|F_2(t, \tilde{U}_N(t))| \le \|\partial_x \tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + c^2_{\alpha,\gamma}\|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda}. \tag{4.28}$$

Thirdly, we use the Cauchy inequality and (3.12) to obtain

$$\begin{aligned}
2\left|\sum_{j=1}^{3} G_j(t, \tilde{U}_N(t))\right| &\le \|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + c(\beta^2 N)^{1-r}(\|F^{-1}_{\alpha,\gamma}\partial_t U(t)\|^2_{H^r_{A,\beta}(\Lambda)} \\
&\quad + (a^2 + c^2_{\alpha,\gamma} + 1)\|F^{-1}_{\alpha,\gamma}U(t)\|^2_{H^r_{A,\beta}(\Lambda)}).
\end{aligned} \tag{4.29}$$

Furthermore, with the aid of the Hölder inequality, (3.23) and (3.12), we verify that

$$\begin{aligned}
&2|G_4(t, \tilde{U}_N(t))| \\
&\quad \le 2a\|U(t) - U_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}(\|U(t)\|_{L^\infty(\Lambda)}\|\tilde{U}_N(t)\|_{\omega_{\alpha,\gamma},\Lambda} + \|U_N(t)\|_{L^4_{\omega_{\alpha,\gamma}}(\Lambda)}\|\tilde{U}_N(t)\|_{L^4_{\omega_{\alpha,\gamma}}(\Lambda)}) \\
&\quad \le 2a\|U(t) - U_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}\|\tilde{U}_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}(\|U(t)\|_{L^\infty(\Lambda)} + c\xi^2_{\alpha,\gamma}\beta^{\frac{1}{2}}N^{\frac{5}{12}}\|U_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}) \\
&\quad \le 2a\|U(t) - U_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}\|\tilde{U}_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}(\|U(t)\|_{L^\infty(\Lambda)} + c\xi^2_{\alpha,\gamma}\beta^{\frac{1}{2}}N^{\frac{5}{12}}(\|U(t)\|_{\omega_{\alpha,\gamma},\Lambda} \\
&\qquad + \|U(t) - U_N(t)\|_{\omega_{\alpha,\gamma},\Lambda})) \\
&\quad \le \|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + 4a^2\|U(t) - U_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda}(\|U(t)\|_{L^\infty(\Lambda)} + c\xi^2_{\alpha,\gamma}\beta^{\frac{1}{2}}N^{\frac{5}{12}}(\|U(t)\|_{\omega_{\alpha,\gamma},\Lambda} \\
&\qquad + \|U(t) - U_N(t)\|_{\omega_{\alpha,\gamma},\Lambda}))^2 \\
&\quad \le \|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + ca^2\beta^{2-2r}N^{\frac{11-6r}{6}}\|F^{-1}_{\alpha,\gamma}U(t)\|^2_{H^r_{A,\beta}(\Lambda)}(N^{-\frac{5}{6}}\|U(t)\|^2_{L^\infty(\Lambda)} \\
&\qquad + \xi^4_{\alpha,\gamma}\beta(\|F^{-1}_{\alpha,\gamma}U(t)\|^2_{H^1_{A,\beta}(\Lambda)} + \|U(t)\|^2_{\omega_{\alpha,\gamma},\Lambda})).
\end{aligned} \tag{4.30}$$

For notational convenience, we set

$$\begin{aligned}
E(\tilde{U}_N, t) &= \|\tilde{U}_N(t)\|^2_{\omega_{\alpha,\gamma},\Lambda} + \int_0^t \|\partial_x \tilde{U}_N(s)\|^2_{\omega_{\alpha,\gamma},\Lambda}ds, \\
d(U) &= 2a + c^2_{\alpha,\gamma} + 2 + 4a\max_{0\le t\le T}\|U(t)\|_{L^\infty(\Lambda)} + ca\xi^2_{\alpha,\gamma}\beta^{\frac{3}{2}}\max_{0\le t\le T}\|F^{-1}_{\alpha,\gamma}U(t)\|_{H^2_{A,\beta}(\Lambda)}, \\
R_r(U, t) &= \left\|F^{-1}_{\alpha,\gamma}\partial_t U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)} + (a^2 + c^2_{\alpha,\gamma} + 1)\left\|F^{-1}_{\alpha,\gamma}U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)} + a^2\|U(t)\|^2_{L^\infty(\Lambda)} \\
&\quad + a^2\xi^4_{\alpha,\gamma}\beta\left(\left\|F^{-1}_{\alpha,\gamma}U(t)\right\|^2_{H^1_{A,\beta}(\Lambda)} + \|U(t)\|^2_{\omega_{\alpha,\gamma},\Lambda}\right).
\end{aligned}$$

In addition,

$$\rho(U_0)=\begin{cases}\|F^{-1}_{\alpha,\gamma}U_0\|^2_{H^r_{A,\beta}(\Lambda)}, & \text{if } u_N(x,0)=\hat{P}_{N,\alpha,\beta,\gamma,\Lambda}U_0(x),\\ 0, & \text{if } u_N(x,0)=\hat{P}^1_{N,\alpha,\beta,\gamma,\Lambda}U_0(x).\end{cases}$$

Thanks to (3.8), we have

$$\|\tilde{U}_N(0)\|^2_{\omega_{\alpha,\gamma},\Lambda}\le c\beta^{2-2r}N^{1-r}\rho(U_0). \tag{4.31}$$

By substituting (4.27)-(4.30) into (4.26), integrating the result with respect to t, and using (4.31), we obtain that

$$\begin{aligned}E(\tilde{U}_N,t)\le&\int_0^t(d(U)E(\tilde{U}_N,s)+ca\xi^2_{\alpha,\gamma}\beta^{\frac12}N^{\frac{5}{12}}E^{\frac32}(\tilde{U}_N,s))ds\\&+c\beta^{2-2r}N^{\frac{11-6r}{6}}\int_0^t R_r(U,s)ds+c\beta^2N^{1-r}\rho(U_0).\end{aligned} \tag{4.32}$$

Proposition 4.2 *(cf. [12]). We suppose that*

(i) $Z(t)$ *is a non-negative function of* t, *and* b_1, b_2 *and* D *are non-negative constants,*
(ii) $D\le\frac{1}{b_2^2}e^{-(b_1+1)t_1}$ *for certain* $t_1>0$,
(iii) for all $t\le t_1$,

$$Z(t)\le\int_0^t(b_1Z(s)+b_2Z^{\frac32}(s))ds+D.$$

Then for all $0\le t\le t_1$,

$$Z(t)\le De^{(b_1+1)t}.$$

Now, let $b_1=d(U)$, $b_2=ca\xi^2_{\alpha,\gamma}\beta^{\frac12}N^{\frac{5}{12}}$, and

$$D=c\beta^{2-2r}N^{\frac{11-6r}{6}}\left(\int_0^t R_r(U,s)ds+N^{-\frac56}\rho(U_0)\right).$$

If integer $r\ge3$ and N is suitably big, then $D\le\frac{1}{b_2^2}e^{-(b_1+1)T}$. Therefore, we use (4.32) and Proposition 4.2 to verify that for all $0\le t\le T$,

$$E(\tilde{U}_N,t)\le c\beta^{2-2r}N^{\frac{11-6r}{6}}e^{(d(U)+1)t}\left(\int_0^t R_r(U,s)ds+N^{-\frac56}\rho(U_0)\right), \tag{4.33}$$

provided that the norms involved at the right side of the above inequality are finite. Finally, the combination of (4.33) and (3.12) leads to that

$$\begin{aligned}&\|U(t)-u_N(t)\|^2_{\omega_{\alpha,\gamma}}+\int_0^t\|\partial_x(U(s)-u_N(s))\|^2_{\omega_{\alpha,\gamma}}ds\\&\le c\beta^{2-2r}N^{\frac{11-6r}{6}}\left(e^{(d(U)+1)t}\left(\int_0^t R_r(U,s)\right)ds+N^{-\frac56}\rho(U_0)\right)\\&\quad+N^{-\frac56}\left(\int_0^t\left\|F^{-1}_{\alpha,\gamma}U(s)\right\|^2_{H^r_{A,\beta}(\Lambda)}ds+\left\|F^{-1}_{\alpha,\gamma}U(t)\right\|^2_{H^r_{A,\beta}(\Lambda)})\right).\end{aligned} \tag{4.34}$$

We next describe the implementation for the spectral scheme (4.23). Let $\phi_l(x)$ be the same as in the last subsection. We expand the numerical solution as

$$u_N(x,t)=\sum_{l=0}^{N}\hat{u}_l(t)\phi_l(x).$$

Inserting the above expression into (4.23) with $\phi=\phi_k(x)$, we obtain

$$\begin{aligned}&\sum_{l=0}^{N}(\phi_l,\phi_k)_{\omega_{\alpha,\gamma},\Lambda}\partial_t\hat{u}_l(t)-a\sum_{l=0}^{N}(\phi_l,\phi_k)_{\omega_{\alpha,\gamma},\Lambda}\hat{u}_l(t)+\sum_{l=0}^{N}(\partial_x\phi_l,\partial_x(\phi_k\omega_{\alpha,\gamma}))_\Lambda\hat{u}_l(t)\\&\quad=(f(t),\phi_k)_{\omega_{\alpha,\gamma},\Lambda}-a\left(u_N^2(t),\phi_k\right)_{\omega_{\alpha,\gamma},\Lambda},\quad 0\le k\le N.\end{aligned}\tag{4.35}$$

We can rewrite the system (4.35) as a compact matrix form. To do this, we introduce the matrices $A=(a_{k,l})_{0\le k,l\le N}$ and $B=(b_{k,l})_{0\le k,l\le N}$, with the following entries:

$$a_{k,l}=(\phi_l,\phi_k)_{\omega_{\alpha,\gamma},\Lambda},\quad b_{k,l}=(\partial_x\phi_l,\partial_x(\phi_k\omega_{\alpha,\gamma}))_\Lambda.$$

Let $\hat{\mathbf{u}}(t)=(\hat{u}_0(t),\hat{u}_1(t),\cdots,\hat{u}_N(t))^T$ and $\mathbf{F}(t)=(F_0(t),F_1(t),\cdots,F_N(t))^T$, with

$$F_k(t)=(f(t),\phi_k)_{\omega_\gamma,\Lambda}-a\left(u_N^2(t),\phi_k\right)_{\omega_{\alpha,\gamma},\Lambda}.$$

Then, the system (4.35) becomes

$$A\partial_t\hat{\mathbf{u}}(t)+(B-aA)\hat{\mathbf{u}}(t)=\mathbf{F}(t).$$

We now consider the Fisher equation (4.19) with $a=0.96$ and $f(x,t)\equiv 0$. For fixedness, we focus on the solutions decaying to zero as $x\to-\infty$, and tending to 1 as $x\to\infty$. In this case, we make the transformation

$$U(x,t)=V(x,t)+\frac{2}{\pi}\arctan\left(\frac{1}{2}e^x\right).$$

Inserting the above expression into (4.19), we obtain the reformed equation for the unknown function $V(x,t)$, vanishing at the infinity. Accordingly, we derive a spectral scheme similar to (4.23). Meanwhile, we adopt the standard explicit fourth-order Runge–Kutta method in time, with the step size τ. The corresponding numerical solution is denoted by $v_{N,\tau}(x,t)$. The numerical solution of the original problem is given by

$$u_{N,\tau}(x,t)=v_{N,\tau}(x,t)+\frac{2}{\pi}\arctan\left(\frac{1}{2}e^x\right).$$

In actual computation, we take the test function (4.20) with $\varepsilon=-1$ and $A=B=1$. We measure the numerical accuracy by the global weighted errors

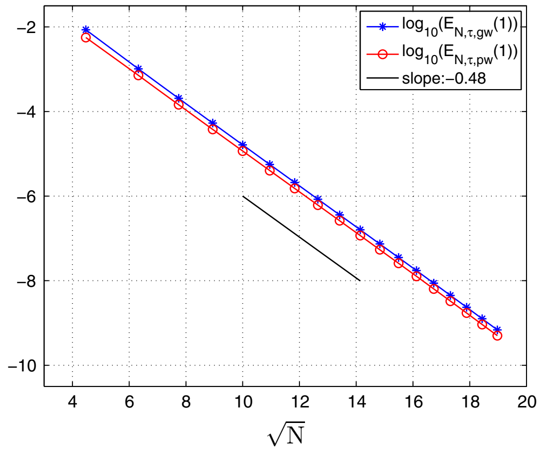
$$E_{N,\tau,gw}(t)=\left(\sum_{j=1}^{N}\left(U\left(\hat{\sigma}_{N,j}^{\beta},t\right)-u_N\left(\hat{\sigma}_{N,j}^{\beta},t\right)\right)^2\hat{\omega}_{N,j}^{\alpha,\beta,\gamma}\right)^{\frac{1}{2}},$$

and the point-wise errors

$$E_{N,\tau,pw}(t)=\max_{0\le j\le N}\left|U\left(\hat{\sigma}_{N,j}^{\beta},t\right)-u_{N,\tau}\left(\hat{\sigma}_{N,j}^{\beta},t\right)\right|.$$

In Fig. 2, we plot the values of the global weighted errors and the point-wise errors at $t=1$, with the parameters $\beta=0.5$ and $\alpha=\gamma=0$, the step size $\tau=0.01$ vs $\sqrt{N}$. They indicate that the numerical errors decay as N increases.

Fig. 2 Comparison of numerical errors with $\tau = 0.01$ and $\beta = 0.5$



5 Concluding Remarks

In this paper, we introduced the orthogonal system of new generalized Hermite functions with the weight function $(1+\frac{2}{\pi}\arctan x)^{-2\alpha}(1-\frac{2}{\pi}\arctan x)^{-2\gamma}$, α and γ being any real numbers. We established the basic results on the corresponding generalized Hermite orthogonal approximation and the related Hermite–Gauss interpolation. By adjusting the parameters α and γ suitably, such approximations might simulate the different asymptotic behaviors of approximated functions at infinities reasonably, and so play important roles in the spectral and pseudospectral methods for various problems with different asymptotic behaviors at infinities. As examples of applications, we provided the spectral schemes for a linear problem and the Fisher equation, and proved their spectral accuracy in space. The numerical results indicated the high efficiency of the suggested algorithms, and coincided well with the analysis. The main idea, the approximation results and the techniques developed in this work are also applicable to other problems defined on the whole line and the related unbounded domains in multiple dimensions.

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