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SOME DEVELOPMENTS IN SPECTRAL METHODS

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Abstract. In this paper, we review some new developments in spectral methods. We first consider the generalized Jacobi spectral method. Then, we present the Jacobi quasi-orthogonal approximation and its applications. Next, we consider the generalized Laguerre spectral method. We also present the Laguerre quasi-orthogonal approximation and its applications.

1. Introduction

The spectral methods have been widely used in scientific computation, see the books of Bernardi and Maday [**BM2**], Canuto, Hussaini, Quarteroni and Zang [**CHQZ1**], Funaro [**F2**], Gottlieb and Orszag [**GO**], and Guo [**G1**], and the references therein.

The traditional spectral methods are available for periodic problems and various problems defined on rectangular domains. Guo [**G2, G3**], and Guo and Wang [**GW1, GW2**] proposed the Jacobi spectral method which is also available for degenerated problems. Later, Guo, Shen and Wang [**GSW1, GSW2**] provided the generalized Jacobi spectral method, which leads to a class of Petrov-Galerkin spectral method for high order problems. It is also appreciated for singular problems. Recently, Guo and Wang [**GWT1, GWT2, GWT3**], and Guo, Sun, and Zhang [**GSZ**] developed the Jacobi quasi-orthogonal approximation, which is very helpful for the Petrov-Galerkin spectral element method of high order problems with mixed inhomogeneous boundary conditions. The Jacobi approximation also serves as an powerful tool for other numerical approaches.

The Laguerre spectral method plays an important role in numerical solutions of various problems defined on unbounded domains. Funaro [**F1**], Guo and Shen [**GS**], Guo and Zhang [**GZX1**], Guo, Wang and Wang [**GWW**], Maday, Pernaud and Vandeven [**MPV**], and Xu and Guo [**XG1**] provided the spectral method by using the generalized Laguerre polynomials. Guo and Ma [**GM**], Guo and Zhang [**GZX2**], and Shen [**S1**] studied the spectral method by using the generalized Laguerre functions. Recently, Guo, Sun, and Zhang [**GSZ**], and Zhang and Guo

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[**ZG**] considered the more general Laguerre approximation and the Laguerre quasi-orthogonal approximation, which are specially appropriate for problems of non-standard type, and exterior problems.

In this paper, we review the recent developments in the Jacobi and Laguerre spectral methods.

2. Jacobi Spectral Method

2.1. Jacobi spectral approximation. Let $\Lambda = \{\, x \mid |x| < 1\}$ and $\chi(x)$ be certain a weight function. For integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ and its norm $||v||_{r,\chi}$ in the usual way. We denote the inner product and the norm of $L^2_\chi(\Lambda)$ by $(u,v)_\chi$ and $||v||_\chi$, respectively.

Let $\alpha, \beta > -1$, and $J_l^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree l. The Jacobi weight function $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. The set of all $J_l^{(\alpha,\beta)}(x)$ is a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system.

For any integer $N \geq 0$, $\mathcal{P}_N(\Lambda)$ stands for the set of all algebraic polynomial of degree at most N. Throughout this paper, we denote by c a generic positive constant independent of any function and N.

The $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathcal{P}_N(\Lambda).$$

If $\partial_x^r v \in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$ and integers $r \geq 0, r \leq N+1$, then

$$||P_{N,\alpha,\beta}v - v||_{\chi^{(\alpha,\beta)}} \leq c_{\alpha,\beta}(N(N+\alpha+\beta))^{-\frac{r}{2}}||\partial_x^r v||_{\chi^{(r+\alpha,r+\beta)}}$$

where $c_{\alpha,\beta}$ is an explicit function of α and β.

In many practical problems, the coefficients of derivatives of different orders may degenerate in different ways. Accordingly, we need certain approximations in non-uniformly weighted spaces. Some approximation results were given by Guo [**G3**], and Guo and Wang [**GW2**]. They play important roles in spectral methods for various problems, whose coefficients might degenerate at the endpoints of Λ. The main strategy is to fit singular solutions by proper Jacobi polynomials, and estimate the errors of numerical solutions in certain non-uniformly weighted Sobolev spaces.

Guo, Wang, Wan and Chu [**GWWC**], and Wang and Guo [**WG1**] also considered other Jacobi orthogonal approximations which are applicable to fourth order problems and multiple-dimensional problems. Shen and Wang [**SW3**] developed the sparse spectral method of high-dimensional problems based on hyperbolic cross.

The Jacobi approximation also serves as an powerful tool for other numerical algorithms. Bernardi and Maday [**BDM**], and Guo and Huang [**GH**] used certain specific Jacobi approximations for axisymmetric and spherically symmetrical domains. Dubiner [**D**], Guo and Wang [**GW4**], Li, Wang, Li and Ma [**LWLM**], and Shen, Wang and Li [**SWL**] considered the spectral method on triangles. Guo and Jia [**GJ1**], and Jia and Guo [**JG**] developed the spectral element method on polygons. Shen and Wang [**SW2**] considered the spectral approximation in elliptic geometries using Mathieu functions.

The Gegenbauer approximation in the weighted Besov space was studied in Babuška and Guo [**BG1, BG2**], which leads to the optimal error estimates of the $p-$version of finite element method for the Poisson equation. The other applications of Gegenbauer approximation to the analysis of finite element method, could be found in Stephan and Suri [**SS**], and the references therein.

2.2. Generalized Jacobi spectral method. Guo, Shen and Wang [**GSW1, GSW2**] proposed the generalized Jacobi orthogonal approximation with arbitrary real parameters α and β. In the sequel, $[\alpha]$ denotes the largest integer $\le \alpha$. Further, $\hat{\alpha} = \bar{\alpha} = -\alpha$ for $\alpha \le -1$, and $\hat{\alpha} = 0, \bar{\alpha} = \alpha$ otherwise. The notations $\hat{\beta}, \bar{\beta}$ and $[\beta]$ have the same meanings.

Let $\bar{l}_{\alpha,\beta} = [\hat{\alpha}] + [\hat{\beta}]$. The generalized Jacobi functions are defined by

$$\bar{J}_l^{(\alpha,\beta)}(x) = \chi^{(\hat{\alpha},\hat{\beta})}(x) J_{l-\bar{l}_{\alpha,\beta}}^{(\bar{\alpha},\bar{\beta})}(x), \qquad l \ge \bar{l}_{\alpha,\beta}.$$

They form a complete $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)-$ orthogonal system. Let

$$Q_{N,\alpha,\beta}(\Lambda) = \text{span}\{\bar{J}_l^{(\alpha,\beta)}(x),\ \bar{l}_{\alpha,\beta} \le l \le N\}.$$

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to Q_{N,\alpha,\beta}(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta} v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in Q_{N,\alpha,\beta}(\Lambda).$$

If $\partial_x^r v \in L^2_{\chi^{(\alpha+r,\beta+r)}}(\Lambda)$, integers $r \ge 1, 0 \le k \le r$, and one of the following conditions holds,

(i) α is negative integer and $\beta > -1$,
(ii) β is negative integer and $\alpha > -1$,
(iii) α and β are negative integers,

then

$$||\partial_x^k (P_{N,\alpha,\beta} v - v)||_{\chi^{(\alpha+k,\beta+k)}} \le c_{\alpha,\beta} N^{k-r} ||\partial_x^r v||_{\chi^{(\alpha+r,\beta+r)}}.$$

The above approximation was used for spectral method of various high order problems, see Guo, Shen and Wang [**GSW1, GSW2**], Shen [**S2**], and Shen and Wang [**SW1**]. Ma and Sun [**MS**] also used a similar trick. The case with $\alpha, \beta \le -1$ is suitable for problems whose coefficients might growing up somewhere.

We now focus on the specific case with $\alpha = -m$ and $\beta = -n$, m and n being any positive integers. For notational convenience, we denote $Q_{N,-m,-n}(\Lambda)$ and $P_{N,-m,-n}v$ by $\bar{Q}_{N,m,n}(\Lambda)$ and $\bar{P}_{N,m,n}v$, respectively. We also introduce the space

$$H^r_{m,n,A}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } ||v||_{H^r_{m,n,A}} < \infty\},$$

equipped with the semi-norm $|v|_{H^r_{m,n,A}} = ||\partial_x^r v||_{\chi^{(-m+r,-n+r)}}$ and the norm $||v||_{H^r_{m,n,A}} = (\sum_{k=0}^{r} |v|^2_{H^k_{m,n,A}})^{\frac{1}{2}}$. Moreover,

$$\begin{aligned} H^r_{0,m,n,A}(\Lambda) &= \{\, v \in H^r_{m,n,A}(\Lambda) \mid \partial_x^k v(-1) = 0 \text{ for } 0 \le k \le n-1, \\ &\qquad \text{and } \partial_x^k v(1) = 0 \text{ for } 0 \le k \le m-1\}, \qquad r \ge \max(m,n). \end{aligned}$$

For integer $\mu \ge \max(m,n)$, the operator $\bar{P}^{\mu,0}_{N,m,n} : H^\mu_{0,m,n,A}(\Lambda) \to \bar{Q}_{N,m,n}(\Lambda)$ is defined by

$$(\partial_x^\mu (v - \bar{P}^{\mu,0}_{N,m,n} v), \partial_x^\mu \phi)_{\chi^{(-m+\mu,-n+\mu)}} = 0, \qquad \forall \phi \in \bar{Q}_{N,m,n}(\Lambda).$$

In fact, $\bar{P}^{\mu,0}_{N,m,n} v = \bar{P}_{N,m,n} v$ for any $v \in H^\mu_{0,m,n,A}(\Lambda)$.

Recently, Guo, Sun, and Zhang [**GSZ**] proposed the generalized Jacobi quasi-orthogonal approximation. To do this, we introduce the following polynomials,

$$q^-_{m,n,j}(x) = \frac{1}{2^m j!}(1-x)^m \sum_{l=0}^{n-1-j} \frac{(m+l-1)!}{2^l l!(m-1)!}(1+x)^{l+j},$$
$$q^+_{m,n,j}(x) = \frac{(-1)^j}{2^n j!}(1+x)^n \sum_{l=0}^{m-1-j} \frac{(n+l-1)!}{2^l l!(n-1)!}(1-x)^{l+j}, \qquad m,n \geq 1.$$

For any $v \in H^\mu_{m,n,A}(\Lambda)$ and $\mu \geq \max(m,n)$, we set

$$v_{m,n,b}(x) = \sum_{j=0}^{n-1} \partial_x^j v(-1) q^-_{m,n,j}(x) + \sum_{j=0}^{m-1} \partial_x^j v(1) q^+_{m,n,j}(x).$$

Furthermore, we let $\bar{v}(x) = v(x) - v_{m,n,b}(x)$. Then, we define the Jacobi quasi-orthogonal projection

$$P^\mu_{*,N,m,n} v(x) = \bar{P}^{\mu,0}_{N,m,n} \bar{v}(x) + v_{m,n,b}(x).$$

It can be shown that $\partial_x^k P^\mu_{*,N,m,n} v(-1) = \partial_x^k v(-1)$ for $0 \leq k \leq n-1$, and $\partial_x^k P^\mu_{*,N,m,n} v(1) = \partial_x^k v(1)$ for $0 \leq k \leq m-1$.

If $v \in H^{\max(m,n)}_{m,n,A}(\Lambda)$, $\partial_x^r v \in L^2_{\chi^{(-m+r,-n+r)}}(\Lambda)$ and integers $m,n,r \geq 1$, $N \geq m+n$, $0 \leq k \leq r \leq N+1$, $\max(m,n,k) \leq \mu \leq m+n$, then

$$||\partial_x^k (P^\mu_{*,N,m,n} v - v)||_{\chi^{(-m+k,-n+k)}} \leq cN^{k-r}(||\partial_x^r v||_{\chi^{(-m+r,-n+r)}} + ||v||_{H^{\max(m,n)}(\Lambda)}).$$

If, in addition, $r \geq m+n$ or $m,n \leq 4$, then

$$||\partial_x^k (P^\mu_{*,N,m,n} v - v)||_{\chi^{(-m+k,-n+k)}} \leq cN^{k-r}||\partial_x^r v||_{\chi^{(-m+r,-n+r)}}.$$

Since the Jacobi quasi-orthogonal approximation fits certain derivatives of approximated functions at the endpoints of Λ, it is very helpful for Petrov-Galerkin spectral element method of high order problems with essential imposition of mixed inhomogeneous Dirichlet-Neumann-Robin boundary conditions, see Guo and Jia [**GJ1**], Guo, Sun, and Zhang [**GSZ**], Guo and Wang [**GWT1, GWT2, GWT3**], Jia and Guo [**JG**], and Wang and Guo [**WTG1, WTG2**].

2.3. Jacobi pseudospectral method. We now turn to the Jacobi pseudospectral method, with which we only need to evaluate the unknown functions on the interpolation nodes, and could deal with nonlinear problems conveniently.

Let $\zeta^{(\alpha,\beta)}_{G,N,j}$, $\zeta^{(\alpha,\beta)}_{R,N,j}$ and $\zeta^{(\alpha,\beta)}_{L,N,j}$, $0 \leq j \leq N$, be the zeros of polynomials $J^{(\alpha,\beta)}_{N+1}(x)$, $(1+x)J^{(\alpha,\beta+1)}_N(x)$ and $(1-x^2)\partial_x J^{(\alpha,\beta)}_N(x)$, respectively. They are arranged in decreasing orders.

The Gauss-type interpolations $\mathcal{I}_{Z,N,\alpha,\beta} v$ are defined by

$$\mathcal{I}_{Z,N,\alpha,\beta} v(\zeta^{(\alpha,\beta)}_{Z,N,j}) = v(\zeta^{(\alpha,\beta)}_{Z,N,j}), \qquad Z = G,R,L,\ 0 \leq j \leq N.$$

Here $Z = G,R,L$ correspond to the Jacobi-Gauss interpolation, the Jacobi-Radau interpolation and the Jacobi-Lobatto interpolation, respectively.

The basic results on the above interpolations could be found in Guo and Wang [**GW2**]. Guo and Zhang [**GZK**] improved them and considered the multiple dimensional cases, which are applicable to second order problems. Wan, Guo and Wang [**WGW**] considered the interpolation, which is related to pseudospectral method of fourth order problems. Some authors developed the pseudospectral element method with its applications, which are also called as spectral method in many literatures, see Bernardi, Maday and Rapetti [**BMR**], Canuto, Hussaini, Quarteroni and Zang [**CHQZ2**], Karniadakis and Sherwin [**KS**], and the references therein.

Recently, Guo, Sun, and Zhang [**GSZ**] proposed the generalized Jacobi-Gauss-Lobatto interpolation.

For integer $r \geq \max(m-1, n-1)$, we define

$$C^r_{0,m,n}(\bar{\Lambda}) = \{v \in C^r(\bar{\Lambda}) \mid \partial^k_x v(-1) = 0 \text{ for } 0 \leq k \leq n-1 \text{ and } \partial^k_x v(1) = 0 \text{ for } 0 \leq k \leq m-1\}.$$

Let

$$\zeta^{(m,n)}_{N,j} = \zeta^{(m,n)}_{G,N-m-n,j}, \qquad m,n \geq 0,\ 0 \leq j \leq N-m-n.$$

For any $v \in C^{\max(m-1,n-1)}_{0,m,n}(\bar{\Lambda})$ and $m, n \geq 1$, we introduce the interpolation $\bar{\mathcal{I}}_{N,m,n} v \in Q_{N,m,n}(\Lambda)$ as follows,

$$\bar{\mathcal{I}}_{N,m,n} v(\zeta^{(m,n)}_{N,j}) = v(\zeta^{(m,n)}_{N,j}), \qquad 0 \leq j \leq N-m-n.$$

Obviously, $\partial^k_x \bar{\mathcal{I}}_{N,m,n} v(-1) = 0$ for $0 \leq k \leq n-1$, and $\partial^k_x \bar{\mathcal{I}}_{N,m,n} v(1) = 0$ for $0 \leq k \leq m-1$.

For any $v \in C^{\max(m-1,n-1)}(\bar{\Lambda})$, we set $\bar{v}(x) = v(x) - v_{m,n,b}(x)$. Then, we define the generalized Jacobi-Gauss-Lobatto interpolation by

$$\bar{\mathcal{I}}_{L,N,m,n} v(x) = \bar{\mathcal{I}}_{N,m,n} \bar{v}(x) + v_{m,n,b}(x).$$

It can be checked that $\partial^k_x \bar{\mathcal{I}}_{L,N,m,n} v(-1) = \partial^k_x v(-1)$ for $0 \leq k \leq n-1$, and $\partial^k_x \bar{\mathcal{I}}_{L,N,m,n} v(1) = \partial^k_x v(1)$ for $0 \leq k \leq m-1$. In addition, $\bar{\mathcal{I}}_{L,N,m,n} v(\zeta^{(m,n)}_{N,j}) = v(\zeta^{(m,n)}_{N,j})$ for $0 \leq j \leq N-m-n$.

If $v \in H^{\max(m,n)}_{m,n,A}(\Lambda)$, $\partial^r_x v \in L^2_{\chi^{(-m+r,-n+r)}}(\Lambda)$ and integers $m, n, r \geq 1$, $N \geq m+n$, $0 \leq k \leq r \leq N+1$, then

$$||\partial^k_x(\bar{\mathcal{I}}_{L,N,m,n} v - v)||_{\chi^{(-m+k,-n+k)}} \leq cN^{k-r}(||\partial^r_x v||_{\chi^{(-m+r,-n+r)}} + ||v||_{H^{\max(m,n)}(\Lambda)}).$$

If, in addition, $1 \leq m, n \leq 4$ or $r \leq m+n$, then

$$||\partial^k_x(\bar{\mathcal{I}}_{L,N,m,n} v - v)||_{\chi^{(-m+k,-n+k)}} \leq cN^{k-r}||\partial^r_x v||_{\chi^{(-m+r,-n+r)}}.$$

The interpolation $\bar{\mathcal{I}}_{L,N,1,1} v$ is the standard Legendre-Gauss-Labatto polynomial interpolation. The corresponding error estimate was first obtained by Guo and Zhang [**GZK**]. If $m = n \geq 1$, then $\bar{\mathcal{I}}_{L,N,m,m} v$ is equivalent to the generalized Legendre-Gauss-Labatto polynomial interpolation $k^m_N v$ given by Bernardi and Maday [**BM2**], in which $||k^1_N v - v||_{\chi^{(-1,-1)}} \leq cN^{-r}||v||_{H^r(\Lambda)}$. Thus, our new results improve and generalize the existing results. The above result also implies $||\partial^m_x(\bar{\mathcal{I}}_{L,N,m,m} v)|| \leq c||\partial^m_x v||$. This means that the interpolation $\bar{\mathcal{I}}_{L,N,m,m} v$ is stable in the space $H^m_{m,m,A}(\Lambda)$.

3. Laguerre Spectral Method

3.1. Generalized Laguerre approximation. Let $\mathcal{R}^+ = (0, \infty)$. The generalized Laguerre weight function $\omega^{(\alpha,\beta)}(x) = x^\alpha e^{-\beta x}, \alpha > -1,\ \beta > 0$. We define the weighted space $H^r_{\omega^{(\alpha,\beta)}}(\mathcal{R}^+)$ and its norm $||v||_{r,\omega^{(\alpha,\beta)}}$ as usual. The inner product and the norm of $L^2_{\omega^{(\alpha,\beta)}}(\mathcal{R}^+)$ are denoted by $(u,v)_{\omega^{(\alpha,\beta)}}$ and $||v||_{\omega^{(\alpha,\beta)}}$, respectively.

Guo and Zhang [**GZX1**] considered the scaled generalized Laguerre polynomials

$$\mathcal{L}_l^{(\alpha,\beta)}(x) = \frac{1}{l!}x^{-\alpha}e^{\beta x}\partial_x^l(x^{l+\alpha}e^{-\beta x}), \qquad l \geq 0.$$

They form a complete $L^2_{\omega^{(\alpha,\beta)}}(\mathcal{R}^+)$–orthogonal system.

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega^{(\alpha,\beta)}}(\mathcal{R}^+) \longrightarrow \mathcal{P}_N(\mathcal{R}^+)$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathcal{P}_N(\mathcal{R}^+).$$

If $\partial_x^r v \in L^2_{\omega^{(\alpha+r,\beta)}}(\mathcal{R}^+)$ and integers $r \geq 0,\ 0 \leq k \leq r \leq N+1$, then

$$\|\partial_x^k(P_{N,\alpha,\beta}v - v)\|_{\omega^{(\alpha+k,\beta)}} \leq c_{\alpha,\beta}(\beta N)^{\frac{k-r}{2}}\|\partial_x^r v\|_{\omega^{(\alpha+r,\beta)}}$$

where the positive constant $c_{\alpha,\beta}$ is given explicitly.

In many practical problems, the coefficients of derivatives of different orders might grow or degenerate in different ways. Thus, we have to consider various orthogonal approximations in certain non-uniformly weighted spaces, see Guo and Zhang [**GZX1**].

We now turn to the generalized Laguerre interpolation. Let $\xi_{Z,N,j}^{(\alpha,\beta)}$ and $\xi_{R,N,j}^{(\alpha,\beta)}$, $0 \leq j \leq N$, be the zeros of $\mathcal{L}_{N+1}^{(\alpha,\beta)}(x)$ and $x\mathcal{L}_N^{(\alpha+1,\beta)}(x)$, respectively. They are arranged in ascending order.

For $v \in C(\bar{\mathcal{R}}^+)$, the generalized Laguerre-Gauss type interpolation $\mathcal{I}_{Z,N,\alpha,\beta}v \in \mathcal{P}_N(\mathcal{R}^+)$ is determined uniquely by

$$\mathcal{I}_{Z,N,\alpha,\beta}v(\xi_{G,N,j}^{(\alpha,\beta)}) = v(\xi_{G,N,j}^{(\alpha,\beta)}), \qquad Z = G, R,\ 0 \leq j \leq N.$$

Here $Z = G, R$ correspond to the generalized Laguerre-Gauss interpolation and the Laguerre-Radau interpolation, respectively.

The approximation errors were estimated in Guo, Wang and Wang [**GWW**]. The special result with $\alpha = 0$ and $\beta = 1$ was first given by Xu and Guo [**XG1**]. Xu and Guo [**XG3**] also investigated the generalized Laguerre approximation in multiple dimensions.

In the early work, we used the Laguerre polynomials mostly, i.e., $\alpha = 0$ and $\beta = 1$. Coulaud, Funaro and Kavian [**CFK**], Funaro [**F1, F2**], Guo and Shen [**GS**], Maday, Pernaud-Thomas and Vandeven [**MPV**], and Xu and Guo [**XG1**] proposed the related spectral and pseudospectral methods for various steady and unsteady problems arising in fluid dynamics and other fields.

Wang and Guo [**WG2**] provided the stair Laguerre spectral method. More precisely, they first used the Laguerre interpolation with moderate mode N, and then adopted the shifted Laguerre interpolation to extend the numerical solutions step by step. Bernardi, Coppoletta and Maday [**BCM**], and Bernardi and Maday [**BM1**] developed the $H^2_{\omega_{0,1}}(\mathcal{R}^+)$-orthogonal approximation and the corresponding interpolation, with their applications. Guo and Xu [**GX**], and Xu and Guo [**XG2**] proposed the mixed Legendre-Laguerre spectral and pseudospectral methods for some problems in an infinite strip.

The generalized Laguerre approximation is more applicable. Guo and Zhang [**GZX1**], and Zhang and Guo [**ZXG**] used it with $\alpha = 2$ in the radius direction, for some problems in the whole three-dimensional space and outside a ball, respectively. Guo and Jiao [**GJ2**], and Guo, Shen and Xu [**GSX**] applied this trick with $\alpha = \beta = 1$, to two-dimensional exterior problems, such as the Navier stokes equations.

3.2. Laguerre approximation using generalized Laguerre functions. Some practical problems are not well-posed in the Laguerre weighted Sobolev spaces. Thus, we often need certain variable transformations for designing reasonable numerical algorithms, which are not convenient in multiple dimensions. On the other hand, the weight functions might destroy some properties of numerical solutions sometimes, which the exact solutions possess. They also bring some difficulties in domain decomposition spectral methods. Thus, if the considered solutions belong to certain Sobolev spaces with the weight function $\tilde{\omega}^{(\alpha)}(x) = x^{\alpha}$, then we prefer to the spectral method using the generalized Laguerre functions.

Guo and Zhang [**GZX2**] considered the scaled generalized Laguerre functions

$$\tilde{\mathcal{L}}_l^{(\alpha,\beta)}(x) = e^{-\frac{1}{2}\beta x}\mathcal{L}_l^{(\alpha,\beta)}(x), \qquad l \geq 0.$$

They form a complete $L^2_{\tilde{\omega}^{(\alpha)}}(\mathcal{R}^+)-$orthogonal system. Let

$$Q_{N,\alpha,\beta}(\mathcal{R}^+) = \{e^{-\frac{1}{2}\beta x}\phi \mid \phi \in \mathcal{P}_N(\mathcal{R}^+)\}.$$

The orthogonal projection $\tilde{P}_{N,\alpha,\beta} : L^2_{\tilde{\omega}^{(\alpha)}}(\mathcal{R}^+) \to Q_{N,\alpha,\beta}(\mathcal{R}^+)$ is defined by

$$(\tilde{P}_{N,\alpha,\beta}v - v, \phi)_{\tilde{\omega}^{(\alpha)}} = 0, \qquad \forall \phi \in Q_{N,\alpha,\beta}(\mathcal{R}^+).$$

If $v \in L^2_{\tilde{\omega}^{(\alpha)}}(\mathcal{R}^+),\ \partial_x^r(e^{\frac{1}{2}\beta x}v) \in L^2_{\omega^{(\alpha+r,\beta)}}(\mathcal{R}^+)$ and integers $r \geq 0, r \leq N+1$, then

$$\|\tilde{P}_{N,\alpha,\beta}v - v\|_{\tilde{\omega}^{(\alpha)}} \leq c_{\alpha,\beta}(\beta N)^{-\frac{r}{2}}||\partial_x^r(e^{\frac{1}{2}\beta x}v)||_{\omega^{(\alpha+r,\beta)}}.$$

In applications, we need several specific orthogonal projections in non-uniformly weighted spaces, which correspond to different underlying problems, see Guo and Zhang [**GZX2**].

Wang, Guo and Wu [**WGW**] studied the related interpolation $\widetilde{\mathcal{I}}_{Z,N,\alpha,\beta}v \in Q_{N,\alpha,\beta}(\mathcal{R}^+)$, determined by

$$\widetilde{\mathcal{I}}_{Z,N,\alpha,\beta}v(\xi^{(\alpha,\beta)}_{Z,N,j}) = v(\xi^{(\alpha,\beta)}_{Z,N,j}), \qquad Z = G, R, \quad 0 \leq j \leq N.$$

Here $z = G, R$ correspond to the generalized Laguerre-Gauss interpolation and the Laguerre-Radau interpolation, respectively. The special case with $\alpha = 0$ and $\beta = 1$ was due to Guo and Wang [**GW3**].

Shen [**S1**] proposed the spectral method using the standard Laguerre functions, i.e., $\alpha = 0, \beta = 1$. At the same time, Guo and Ma [**GM**], and Ma and Guo [**MG**] used such basis functions for domain decomposition methods. If the solutions vary fast or have several peaks, then we could refine the numerical results by the multidomain Legendre approximation between the Laguerre interpolation nodes. This technique is suitable for parallel computation, and recovers the structures of approximated solutions between the interpolation nodes, see Guo and Wang [**GW3**]. Besides, Shen and Wang [**SW1**] proposed the composite Legendre-Laguerre dual-Petrov-Galerkin methods for third-order equations.

The generalized Laguerre functions are powerful tools for a number of important problems. For instance, we consider the Black-Scholes type equation

$$\begin{cases} \partial_s V(x,s) + xB\partial_x V(x,s) + \partial_x(x^2 A\partial_x V(x,s)) - GV(x,s) = F(x,s), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathcal{R}^+,\ 0 \leq s < T, \\ V(x,T) = V_0(x), \qquad\qquad\qquad\qquad\qquad\quad x \in \mathcal{R}^+ \end{cases}$$

where $0 < a_0 \leq A(x,s) \leq a_1$, and $|B(x,s)|$ is bounded as $x \to 0$. According to Fichera's theory, we could not impose any boundary condition at $x = 0$. This equation includes many models in financial mathematics, such as the Black-Scholes model, the Dothan model and the Black-Derman-Toy model. Guo and Zhang [**GZX2**] proposed the spectral method for solving the put-options which decay to zero rapidly as $x \to \infty$.

Zhang, Wang and Guo [**ZWG**], and Wang, Guo and Zhang [**WGZ**] used the generalized Laguerre approximation, coupled with the Fourier approximation and the spherical harmonic approximation, for two and three dimensional exterior problems, respectively.

3.3. Laguerre approximation with arbitrary parameter α. Guo, Sun, and Zhang [**GSZ**] proposed the generalized Laguerre approximation with arbitrary real parameter α. Denote by $[\alpha]$ the largest integer $\leq \alpha$. Let $\bar{l}_\alpha = [-\alpha]$ for $\alpha \leq -1$, and $\bar{l}_\alpha = 0$ for $\alpha > -1$. Meanwhile, $l_\alpha = l - [-\alpha]$ for $\alpha \leq -1$, and $l_\alpha = l$ for $\alpha > -1$.

The new generalized Laguerre functions are defined by

$$\bar{\mathcal{L}}_l^{(\alpha,\beta)}(x) = \begin{cases} x^{-\alpha}\mathcal{L}_{l_\alpha}^{(-\alpha,\beta)}(x), & \alpha \leq -1,\ l \geq \bar{l}_\alpha, \\ \mathcal{L}_l^{(\alpha,\beta)}(x), & \alpha > -1,\ l \geq \bar{l}_\alpha. \end{cases}$$

They conform a $L^2_{\omega^{(\alpha,\beta)}}(\mathcal{R}^+)-$orthogonal system. Let

$$Q_{N,\alpha,\beta}(\mathcal{R}^+) = \text{span}\{\ \bar{\mathcal{L}}_l^{(\alpha,\beta)}(x),\ \bar{l}_\alpha \leq l \leq N\}.$$

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega^{(\alpha,\beta)}}(\mathcal{R}^+) \to Q_{N,\alpha,\beta}(\mathcal{R}^+)$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in Q_{N,\alpha,\beta}(\mathcal{R}^+).$$

The approximation error was estimated by Guo, Sun, and Zhang [**GSZ**]. In practice, the specific projection $P_{N,-m,\beta}v$ is the most useful, m being any positive integer. Everitt, Littlejohn and Wellman [**ELW**] also considered such approximation without the error estimate. For simplicity, we denote $Q_{N,-m,\beta}(\mathcal{R}^+)$ and $P_{N,-m,\beta}v$, by $\bar{Q}_{N,m,\beta}(\mathcal{R}^+)$ and $\bar{P}_{N,m,\beta}v$, respectively.

For numerical solutions of high order differential equations, we need other orthogonal projections. For this purpose, we introduce the following space with integer $r \geq 0$,

$$H^r_{m,\beta,A}(\mathcal{R}^+) = \{\ v|\ v \text{ is measurable on } \mathcal{R}^+ \text{ and } \|v\|_{H^r_{m,\beta,A}} < \infty\},$$

equipped with the semi-norm $|v|_{H^r_{m,\beta,A}} = \|\partial_x^r v\|_{\omega^{(-m+r,\beta)}}$ and the norm $\|v\|_{H^r_{m,\beta,A}} = (\sum_{k=0}^{r} \|\partial_x^k v\|^2_{\omega^{(-m+k,\beta)},A})^{\frac{1}{2}}$. Moreover,

$${}_0H^r_{m,\beta,A}(\mathcal{R}^+) = \{\ v \mid v \in H^r_{m,\beta,A}(\mathcal{R}^+) \text{ and } \partial_x^k v(0) = 0 \text{ for } 0 \leq k \leq r-1\},$$

$$B^m_{m,\beta}(\mathcal{R}^+) = {}_0H^m_{m,\beta,A}(\mathcal{R}^+) \cap H^r_{m,\beta,A}(\mathcal{R}^+), \qquad 1 \leq m \leq r.$$

For integer $m \geq 1$, the operator ${}_0\bar{P}^m_{N,m,\beta} : B^m_{m,\beta}(\mathcal{R}^+) \to \bar{Q}_{N,m,\beta}(\mathcal{R}^+)$ is defined by

$$(\partial_x^m(v - {}_0\bar{P}^m_{N,m,\beta}v),\ \partial_x^m\phi)_{\omega^{(0,\beta)}} = 0, \qquad \forall\ \phi \in \bar{Q}_{N,m,\beta}(\mathcal{R}^+).$$

In fact, ${}_0\bar{P}^m_{N,m,\beta}v(x) = \bar{P}_{N,m,\beta}v(x)$ for any $v \in B^m_{m,\beta}(\mathcal{R}^+)$.

We now turn to the Laguerre quasi-orthogonal approximation. Let

$$v_{b,m}(x) = \sum_{j=0}^{m-1} \partial_x^j v(0)\frac{x^j}{j!}.$$

For any $v \in H^m_{m,\beta,A}(\mathcal{R}^+)$ with $\mu \geq m$, we set $\bar{v}(x) = v(x) - v_{b,m}(x)$. Then, we define the Laguerre quasi-orthogonal projection by

$$P^m_{*,N,m,\beta}v(x) = {}_0\bar{P}^m_{N,m,\beta}\bar{v}(x) + v_{b,m}(x) \in \mathcal{P}_N(\mathcal{R}^+).$$

Obviously, $\partial^k_x P^m_{*,N,-m,\beta}v(0) = \partial^k_x v(0)$ for $0 \leq k \leq m-1$.

If $v \in H^m_{m,\beta,A}(\mathcal{R}^+)$, $\partial^r_x v \in L^2_{\omega^{(-m+r,\beta)}}(\mathcal{R}^+)$ and integers $1 \leq m \leq \min(r,N)$, $0 \leq k \leq r \leq N+1$, $k \leq m$, then

$$\|\partial^k_x(P^m_{*,N,m,\beta}v \quad -v)\|_{\omega^{(-m+k,\beta)}} \leq c(\beta N)^{\frac{k-r}{2}}\|\partial^r_x v\|_{\omega^{(-m+r,\beta)}}.$$

Guo, Sun, and Zhang [**GSZ**] also considered the corresponding interpolation. For integer $r \geq m-1$, we set

$${}_0C^r_m(\bar{\mathcal{R}}^+) = \{v \in C^r(\bar{\mathcal{R}}^+) |\ \partial^k_x u(0) = 0,\ 0 \leq k \leq m-1\}.$$

Let

$$\xi^{(m,\beta)}_{N,j} = \xi^{(m,\beta)}_{G,N-m,j}, \qquad 0 \leq j \leq N-m,\ m \geq 1.$$

For any $v \in {}_0C^{m-1}_m(\bar{\mathcal{R}}^+)$, we introduce the interpolation

$$\bar{\mathcal{I}}_{N,m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j}), \qquad 0 \leq j \leq N-m.$$

For any $v \in C^{m-1}_m(\bar{\mathcal{R}}^+)$, we set $\bar{v}(x) = v(x) - v_{b,m}(x)$. Then, we define the new interpolation by

$$\bar{\mathcal{I}}_{R,N,m,\beta}v(x) = \bar{\mathcal{I}}_{N,m,\beta}\bar{v}(x) + v_{b,m}(x).$$

It can be checked that $\bar{\mathcal{I}}_{R,N,m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j})$ for $0 \leq j \leq N-m$. In addition, $\bar{\mathcal{I}}_{R,N,m,\beta}v(0) = \partial^k_x v(0)$ for $0 \leq k \leq m-1$. Actually, this interpolation is the same as the generalized Laguerre-Gauss-Radau interpolation.

If $v \in H^m_{m,\beta,A}(\mathcal{R}^+)$, $\partial^r_x v \in L^2_{\omega^{(-m+r,\beta)}}(\mathcal{R}^+)$ and integers $1 \leq m \leq \min(r,N)$, $0 \leq k \leq r \leq N+1$, $k \leq m$, then

$$\|\partial^k_x(\bar{\mathcal{I}}_{R,N,m,\beta}v - v)\|_{\omega^{(-m+k,\beta)}} \leq c(\beta^{-\frac{1}{2}}+1)(\ln N)^{\frac{1}{2}}(\beta N)^{\frac{k+1-r}{2}}\|\partial^r_x v\|_{\omega^{(-m+r,\beta)}}.$$

This result improves and generalizes the existing results essentially.

Based on the previous approximations, Guo, Sun, and Zhang [**GSZ**] designed the Petrov-Galerkin spectral and collection methods with essential imposition of mixed inhomogeneous boundary conditions, for some high order differential equations similar to the steady beam equation and the steady extended Fisher-Kolmogorov equation.

3.4. Laguerre approximation using Laguerre functions with arbitrary parameter α. Zhang and Guo [**ZG**] proposed the generalized Laguerre approximation using the Laguerre functions with arbitrary parameter α, which are given by

$$\hat{\mathcal{L}}^{(\alpha,\beta)}_l(x) = e^{-\frac{\beta}{2}x}\bar{\mathcal{L}}^{(\alpha,\beta)}_l(x), \qquad l \geq \bar{l}_\alpha.$$

They form a complete $L^2_{\tilde{\omega}^{(\alpha)}}(\mathcal{R}^+)$−orthogonal system. Let

$$\tilde{Q}_{N,\alpha,\beta}(\mathcal{R}^+) = \{e^{-\frac{\beta}{2}x}\phi \mid \phi \in Q_{N,\alpha,\beta}(\mathcal{R}^+)\}.$$

The $L^2_{\tilde{\omega}^{(\alpha)}}(\mathcal{R}^+)$-orthogonal projection $\tilde{P}_{N,\alpha,\beta} : L^2_{\tilde{\omega}^{(\alpha)}}(\mathcal{R}^+) \to \tilde{Q}_{N,\alpha,\beta}(\mathcal{R}^+)$, is defined by

$$(\tilde{P}_{N,\alpha,\beta,\Lambda}v - v, \phi)_{\tilde{\omega}^{(\alpha)}} = 0, \qquad \forall \phi \in \tilde{Q}_{N,\alpha,\beta}(\mathcal{R}^+).$$

The approximation error was estimated in Zhang and Guo [**ZG**]. In practice, the specific projection $\tilde{P}_{N,-m,\beta}v$ is the most useful, m being any positive integer.

We now introduce the following space with integer $r \geq 0$,

$$H^r_{m,A}(\mathcal{R}^+) = \{\ v|\ v \text{ is measurable on } \mathcal{R}^+ \text{ and } \|v\|_{H^r_{m,A}} < \infty\},$$

with the semi-norm $|v|_{H^r_{m,A}} = ||\partial_x^r v||_{\tilde{\omega}^{(-m+r)}}$ and the norm $\|v\|_{H^r_{m,A}} = (\sum_{k=0}^{r} |v|^2_{H^r_{m,A}})^{\frac{1}{2}}$. Moreover,

$${}_0H^r_{m,A}(\mathcal{R}^+) = \{\ v \mid v \in H^r_{m,A}(\mathcal{R}^+) \text{ and } \partial_x^k v(0) = 0 \text{ for } 0 \leq k \leq r-1\},\ 1 \leq m \leq r.$$

Let $\hat{Q}_{N,m,\beta}(\mathcal{R}^+) = \tilde{Q}_{N,-m,\beta}(\mathcal{R}^+)$, and ${}_0\bar{P}^m_{N,m,\beta}v$ be the same as in the last subsection. If $v \in {}_0H^m_{m,A}(\mathcal{R}^+)$, then ${}_0\bar{P}^m_{N,m,\beta}(e^{\frac{\beta}{2}x}v)$ is meaningful.

For any $v \in H^m_{m,A}(\mathcal{R}^+)$, we set $\hat{v}(x) = v(x) - v_{b,m}(x)$, with

$$v_{b,m}(x) = e^{-\frac{\beta}{2}x} \sum_{j=0}^{m-1} \big(\frac{1}{j!} \sum_{i=0}^{j} C^i_j (\frac{\beta}{2})^{j-i} \partial_x^i v(0)\big) x^j.$$

Then, we define the Laguerre quasi-orthogonal projection by

$$P^m_{*,N,m,\beta}v(x) = e^{-\frac{\beta}{2}x}{}_0\bar{P}^m_{N,m,\beta}(e^{\frac{\beta}{2}x}\hat{v}(x)) + v_{b,m}(x).$$

Evidently, $P^m_{*,N,m,\beta}v \in \hat{Q}_{N,m,\beta}(\mathcal{R}^+)$. Moreover, $\partial_x^k P^m_{*,N,m,\beta}v(0) = \partial_x^k v(0)$ for $0 \leq k \leq m-1$.

If $v \in H^m_{m,A}(\mathcal{R}^+) \cap H^k_{\tilde{\omega}^{(-m+k)}}(\mathcal{R}^+)$, $\partial_x^r(e^{\frac{\beta}{2}x}v) \in L^2_{\omega^{(-m+r,\beta)}}(\Lambda)$ and integers $0 \leq k \leq m \leq \min(r,N) \leq N+1$, then

$$\|\partial_x^k(P^m_{*,N,m,\beta}v - v)\|_{\tilde{\omega}^{(-m+k)}} \leq c(\beta N)^{\frac{k-r}{2}} \|\partial_x^r(e^{\frac{\beta}{2}x}v)\|_{\omega^{(-m+r,\beta)}}.$$

We now turn to the corresponding interpolation. For any $v \in {}_0C^{m-1}_m(\mathcal{R}^+)$ and $m \geq 1$, the new interpolation $\hat{\mathcal{I}}_{N,m,\beta}v \in \hat{Q}^{(m,\beta)}_N(\bar{\mathcal{R}}^+)$ is determined uniquely by

$$\hat{\mathcal{I}}_{N,m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j}), \qquad 0 \leq j \leq N-m.$$

For any $v \in C^{m-1}_m(\bar{\mathcal{R}}^+)$, we set $\hat{v}(x) = v(x) - v_{b,m}(x)$. Then, we define the new Lagurre-Gauss-Radau type interpolation as

$$\hat{\mathcal{I}}_{R,N,m,\beta}v(x) = \hat{\mathcal{I}}_{N,m,\beta}\hat{v}(x) + v_{b,m}(x).$$

It can be checked that $\hat{\mathcal{I}}_{R,N,m,\beta}v(\xi^{(m,\beta)}_{N,j}) = v(\xi^{(m,\beta)}_{N,j})$ for $0 \leq j \leq N-m$. In addition, $\partial_x^k \hat{\mathcal{I}}_{R,N,m,\beta,\Lambda}v(0) = \partial_x^k v(0)$ for $0 \leq k \leq m-1$.

If $v \in H^m_{m,A}(\Lambda) \cap H^k_{\tilde{\omega}^{(-m+k)}}(\mathcal{R}^+), \partial_x^r(e^{\frac{\beta}{2}x}v) \in L^2_{\omega^{(-m+r,\beta)}}(\mathcal{R}^+)$ and integers $0 \leq k \leq m \leq \min(r,N) \leq N+1$, then

$$\|\partial_x^k(\hat{\mathcal{I}}_{R,N,m,\beta}v - v)\|_{\tilde{\omega}^{(-m+k)}} \leq c(\beta^{-\frac{1}{2}}+1)(\ln N)^{\frac{1}{2}}(\beta N)^{\frac{k+1-r}{2}}||\partial_x^r(e^{\frac{\beta}{2}x}v)||_{\omega^{(-m+r,\beta)}}.$$

The above approach is very helpful for solving problems of non-standard type. For example, we consider the Fokker-Planck equation with the positive parameters β and μ, describing the Brownian motion of particles in an infinite channel,

$$\begin{cases} \partial_t U(x,y,t) + x\partial_y U(x,y,t) - \beta\partial_x(xU(x,y,t)) + y\partial_x U(x,y,t) & \\ \quad -\beta\mu\partial_x^2 U(x,y,t) = 0, & x \in \mathcal{R},\ |y| < 1,\ 0 < t \leq T, \\ U(x,y,t) = 0, & x \geq 0, y = -1 \text{ or } x \leq 0, y = 1,\ 0 < t \leq T, \\ U(x,y,t) \to 0, & |x| \to \infty,\ |y| < 1, 0 < t \leq T, \\ U(x,y,0) = U_0(x,y), & x \in \mathcal{R}, |y| \leq 1. \end{cases}$$

There exist several difficulties for solving the above problem numerically. Firstly, it behaves like parabolic equation in the x-direction, and like hyperbolic equation in the y-direction. Next, the coefficient of the convective term $\partial_y U$ changes the sign at $x = 0$, and so different kinds of boundary conditions should imposed on the different subdomains with $x < 0$ or $x > 0$, respectively. Thirdly, the terms $\partial_x U(x, y, t)$ and $\partial_y U(x, y, t)$ possess the coefficient x varying from $-\infty$ to ∞. Guo and Wang [**GWT2**] introduced a proper composite approximation, which is a set of two mixed generalized Laguerre-Legendre quasi-orthogonal approximations on the subdomains. The numerical solution of the corresponding spectral method keeps the continuity at $x = 0$, and the global spectral accuracy. We also refer to Wang and Guo [**WTG1**] for the pseudospectral method.

Another challenging problem is how to design reasonable spectral and pseudospectral methods for exterior problems with polygon obstacles. Guo and Wang [**GWT2, GWT3**], and Wang and Guo [**WTG2**] divided the exterior domain into several subdomains, and constructed different quasi-orthogonal approximations on different subdomains. These approximations form a composite approximation on the whole exterior domain, keeping certain continuity on the interfaces of adjacent subdomains and possessing the global spectral accuracy. The related algorithms provided accurate numerical results of second and fourth order problems.

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