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The spectral method for high order problems with proper simulations of asymptotic behaviors at infinity

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ABSTRACT

In this paper, we investigate new spectral and multidomain spectral methods for high order problems. We introduce a family of new generalized Laguerre functions, which are mutually orthogonal with the weight function $x^{\alpha}(\delta+x)^{-\gamma}$, $\delta > 0$, α and γ being arbitrary real numbers. The corresponding quasi-orthogonal approximation and Laguerre–Gauss–Radau type interpolation are proposed. The spectral and multidomain spectral schemes are provided for several model problems, which not only fit the mixed inhomogeneous boundary conditions on the fixed boundary exactly, but also match the asymptotic behaviors at infinity reasonably. Numerical results demonstrate the efficiency of suggested algorithms, and confirm the analysis well.

1. Introduction

The spectral and pseudospectral methods have been used successfully for scientific computation; see [1–8] and the references therein. During the past decade, more and more attention was paid to numerical solutions of differential equations defined on unbounded domains. We often used the Laguerre spectral and pseudospectral methods for various problems defined on the half line and the related unbounded domains, as well as certain exterior problems; see, e.g., [9–15].

The mathematical foundations of the Laguerre spectral and pseudospectral methods are the Laguerre orthogonal approximation and the Laguerre–Gauss–Radau interpolation. Funaro [9], Guo and Shen [10], and Maday et al. [13] developed the standard Laguerre orthogonal approximation with its applications to nonlinear partial differential equations. Shen [14] studied the orthogonal approximation using the Laguerre functions. This is more suitable for many practical problems, since the weight function equals 1 in their weak formulations. Guo and Ma [16] used such approximation for the related multidomain method. Later, Guo et al. [17], and Guo and Zhang [18] considered the orthogonal approximations by using the generalized Laguerre polynomials and functions respectively, which were applied to some important partial differential equations with coefficients growing up at infinity, such as the Fokker–Planck equation and the Black–Scholes type equations; see [18,19]. They are also efficient tools for dealing with certain exterior problems. However, the above approaches are

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available for second order problems essentially. Recently, Guo et al. [20], and Zhang and Guo [21] proposed the two kinds of Laguerre quasi-orthogonal approximations and the related interpolations, which are helpful for numerical solutions of high order problems. The first one is especially available for solutions growing up at infinity, while the second one is suitable for solutions decaying like $(1+x)^{\mu}$, $\mu < -\frac{1}{2}$. Whereas, in practical cases, the solutions might decay in different ways. Generally speaking, if the solution behaves like $\mathcal{O}((1+x)^{\mu})$ as x increases, then it seems better to adopt the orthogonal approximation with the weight function like $(1+x)^{-\gamma}$, $\gamma > 2\mu+1$.

In this paper, we investigate the new Laguerre spectral and multidomain spectral methods for high order problems with various mixed inhomogeneous boundary conditions and asymptotic behaviors. We shall introduce the new Laguerre functions, which are mutually orthogonal with the weight function $x^{\alpha}(\delta+x)^{-\gamma}$, $\delta > 0$, α and γ being any real numbers. Then, we study the corresponding orthogonal approximation. By adjusting the parameters α and γ suitably, this approach not only fits the boundary conditions of considered functions at the fixed boundary exactly, but also simulates their asymptotic behaviors at infinity reasonably. We next propose the related quasi-orthogonal approximation, with which it is more convenient to deal with mixed inhomogeneous boundary conditions. Moreover, it plays an important role in the multidomain spectral method for high order problems. We also propose the related Laguerre–Gauss–Radau type interpolation, which serves as the mathematical foundation of new Laguerre pseudospectral and collocation methods. As examples of applications, we provide the spectral and multidomain spectral methods for several model problems, and prove their spectral accuracy. The numerical results demonstrate the efficiency of suggested algorithms, and coincide well with the analysis.

This paper is organized as follows. The next section is for preliminaries. In Section 3, we propose the new Laguerre orthogonal and quasi-orthogonal approximations. In Section 4, we study the related Laguerre–Gauss–Radau type interpolation. In Section 5, we provide the spectral and multidomain spectral schemes for model problems. In Section 6, we present some numerical results. The final section is for concluding remarks.

2. Preliminaries

In this section, we recall some recent results on the generalized Laguerre orthogonal approximation and the Laguerre–Gauss–Radau type interpolation.

Let $\Lambda = \{x \mid 0 < x < \infty\}$ and $\chi(x)$ be certain a weight function. For integer $r \geq 0$, we define the weighted Hilbert space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(\cdot,\cdot)_{r,\chi,\Lambda}$, the semi-norm $|\cdot|_{r,\chi,\Lambda}$ and the norm $\|\cdot\|_{r,\chi,\Lambda}$. In particular, the inner product and the norm of $L^2_\chi(\Lambda)$ are denoted by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$, respectively. For simplicity, we denote $\frac{d^k v}{dx^k}$ by $\partial_x^k v$. For integer $r \geq 1$,

$$ {}_0H^r_\chi(\Lambda) = \{v \in H^r_\chi(\Lambda) \mid \partial_x^k v(0) = 0,\ 0 \leq k \leq r-1\}. $$

We omit the subscript χ in notations whenever $\chi(x) \equiv 1$.

The scaled generalized Laguerre polynomials of degree $l \geq 0$ are given by (cf. [22])

$$ L_l^{(\alpha,\beta)}(x) = \frac{1}{l!} x^{-\alpha} e^{\beta x} \partial_x^l (x^{l+\alpha} e^{-\beta x}), \quad \alpha > -1,\ \beta > 0,\ l \geq 0. $$

Now, let α be any real number. Denote by $[\alpha]$ the largest integer $\leq \alpha$. Let $\bar{l}_\alpha = [-\alpha]$ for $\alpha \leq -1$, and $\bar{l}_\alpha = 0$ for $\alpha > -1$. Meanwhile, $l_\alpha = l - [-\alpha]$ for $\alpha \leq -1$, and $l_\alpha = l$ for $\alpha > -1$. Guo et al. [20] introduced the following functions with real parameter α,

$$ \mathcal{L}_l^{(\alpha,\beta)}(x) = \begin{cases} x^{-\alpha} L_{l_\alpha}^{(-\alpha,\beta)}(x), & \alpha \leq -1,\ l \geq \bar{l}_\alpha = [-\alpha], \\ L_l^{(\alpha,\beta)}(x), & \alpha > -1,\ l \geq \bar{l}_\alpha = 0. \end{cases} \tag{2.1} $$

They are the eigenfunctions of the following Sturm–Liouville problem (see (4.6) of [20]),

$$ \partial_x(x^{\alpha+1} e^{-\beta x} \partial_x \mathcal{L}_l^{(\alpha,\beta)}(x)) + \lambda_l^{(\alpha,\beta)} x^{\alpha} e^{-\beta x} \mathcal{L}_l^{(\alpha,\beta)}(x) = 0, \quad l \geq \bar{l}_\alpha, \tag{2.2} $$

with the eigenvalues

$$ \lambda_l^{(\alpha,\beta)} = \begin{cases} \beta(l_\alpha - \alpha) = \beta(l - [-\alpha] - \alpha), & \text{for } \alpha \leq -1, \\ \beta l_\alpha = \beta l, & \text{for } \alpha > -1. \end{cases} \tag{2.3} $$

Next, let $\omega_{\alpha,\beta}(x) = x^{\alpha} e^{-\beta x}$, and

$$ \gamma_l^{(\alpha,\beta)} = \frac{\Gamma(l+\alpha+1)}{\beta^{\alpha+1} l!}, \quad \alpha > -1, \beta > 0,\ l \geq 0. $$

Due to (4.8) of [20], we have

$$ \int_\Lambda \mathcal{L}_l^{(\alpha,\beta)}(x) \mathcal{L}_{l'}^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) dx = \eta_l^{(\alpha,\beta)} \delta_{l,l'} \tag{2.4} $$

where $\delta_{l,l'}$ is the Kronecker symbol, and

$$\eta_l^{(\alpha,\beta)} = \begin{cases} \gamma_{l_\alpha}^{(-\alpha,\beta)} = \gamma_{l-[-\alpha]}^{(-\alpha,\beta)}, & \text{for } \alpha \le -1,\, l \ge \bar{l}_\alpha, \\ \gamma_l^{(\alpha,\beta)}, & \text{for } \alpha > -1,\, l \ge 0. \end{cases} \tag{2.5}$$

The set of all $\mathcal{L}_l^{(\alpha,\beta)}(x)$ is complete in the space $L^2_{\omega_{\alpha,\beta}}(\Lambda)$. Hence, for any $v \in L^2_{\omega_{\alpha,\beta}}(\Lambda)$, we have

$$v(x) = \sum_{l=\bar{l}_\alpha}^{\infty} \hat{v}_l^{(\alpha,\beta)} \mathcal{L}_l^{(\alpha,\beta)}(x) \tag{2.6}$$

with

$$\hat{v}_l^{(\alpha,\beta)} = \frac{1}{\eta_l^{(\alpha,\beta)}} \int_\Lambda v(x) \mathcal{L}_l^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) dx. \tag{2.7}$$

For integer $N \ge \bar{l}_\alpha$, we set

$$Q_N^{(\alpha,\beta)}(\Lambda) = \mathrm{span}\{\mathcal{L}_{\bar{l}_\alpha}^{(\alpha,\beta)}(x), \mathcal{L}_{\bar{l}_\alpha+1}^{(\alpha,\beta)}(x), \ldots, \mathcal{L}_N^{(\alpha,\beta)}(x)\}.$$

The orthogonal projection $P_{N,\alpha,\beta,\Lambda} : L^2_{\omega_{\alpha,\beta}}(\Lambda) \to Q_N^{(\alpha,\beta)}(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta,\Lambda} v - v, \phi)_{\omega_{\alpha,\beta},\Lambda} = 0, \quad \forall \phi \in Q_N^{(\alpha,\beta)}(\Lambda).$$

In practice, the specific case with $\alpha = -m$ is the most useful. According to Theorem 4.2 of [20], we have the following result.

Lemma 2.1. *If $\partial_x^\nu v \in L^2_{\omega_{-m+\nu,\beta}}(\Lambda)$ for $\nu = 0, k, r$, and integers $1 \le m \le \min(r, N)$, then for $0 \le k \le r \le N+1$,*

$$\left\| \partial_x^k \left(P_{N,-m,\beta,\Lambda} v - v \right) \right\|_{\omega_{-m+k,\beta},\Lambda} \le c(\beta N)^{\frac{k-r}{2}} \| \partial_x^r v \|_{\omega_{-m+r,\beta},\Lambda}. \tag{2.8}$$

Hereafter, we denote by c a generic positive constant independent of any function and N.

Next, we introduce the following space with integer $r \ge 0$,

$$H^r_{\omega_{-m,\beta},A}(\Lambda) = \left\{ v \mid v \text{ is measurable on} \Lambda \text{ and } \|v\|_{H^r_{\omega_{-m,\beta},A}(\Lambda)} < \infty \right\},$$

equipped with the semi-norm and the norm as

$$|v|_{H^r_{\omega_{-m,\beta},A}(\Lambda)} = \|\partial_x^r v\|_{\omega_{-m+r,\beta},\Lambda}, \qquad \|v\|_{H^r_{\omega_{-m,\beta},A}(\Lambda)} = \left(\sum_{k=0}^r |v|^2_{H^k_{\omega_{-m,\beta},A}(\Lambda)} \right)^{\frac{1}{2}}.$$

Moreover, for $1 \le m \le r$,

$${}_0H^r_{\omega_{-m,\beta},A}(\Lambda) = \{v \mid v \in H^r_{\omega_{-m,\beta},A}(\Lambda) \text{ and } \partial_x^k v(0) = 0 \text{ for } 0 \le k \le r-1\}.$$

The projection ${}_0P^m_{N,-m,\beta,\Lambda} : {}_0H^m_{\omega_{-m,\beta},A}(\Lambda) \to Q_N^{(-m,\beta)}(\Lambda)$ is defined by

$$(\partial_x^m({}_0P^m_{N,-m,\beta,\Lambda} v - v),\ \partial_x^m \phi)_{\omega_{0,\beta},\Lambda} = 0, \quad \forall\, \phi \in Q_N^{(-m,\beta)}(\Lambda).$$

In fact, ${}_0P^m_{N,-m,\beta,\Lambda} v = P_{N,-m,\beta,\Lambda} v$ for any $v \in {}_0H^r_{\omega_{-m,\beta},A}(\Lambda)$. Therefore, (2.8) implies the following result.

Lemma 2.2. *If $v \in {}_0H^m_{\omega_{-m,\beta},A}(\Lambda)$, $\partial_x^r v \in L^2_{\omega_{-m+r,\beta}}(\Lambda)$, integers $1 \le m \le \min(r, N)$ and $0 \le k \le \min(m, r)$, then for $0 \le r \le N+1$,*

$$\| \partial_x^k \left({}_0P^m_{N,-m,\beta,\Lambda} v - v \right) \|_{\omega_{-m+k,\beta},\Lambda} \le c(\beta N)^{\frac{k-r}{2}} \| \partial_x^r v \|_{\omega_{-m+r,\beta},\Lambda}. \tag{2.9}$$

We now turn to the Laguerre–Gauss–Radau type interpolation. For $\alpha > -1$ and $\beta > 0$, we denote by $\xi^{(\alpha,\beta)}_{G,N,j,\Lambda} (0 \le j \le N)$ the zeros of the polynomial $L^{(\alpha,\beta)}_{N+1}(x)$, which are arranged in ascending order. Meanwhile, $\omega^{(\alpha,\beta)}_{G,N,j,\Lambda} (0 \le j \le N)$ stand for the corresponding Christoffel numbers such that for any polynomial $\phi(x)$ of degree at most $2N+1$,

$$\int_\Lambda \phi(x) \omega_{\alpha,\beta}(x) dx = \sum_{j=0}^N \phi(\xi^{(\alpha,\beta)}_{G,N,j,\Lambda}) \omega^{(\alpha,\beta)}_{G,N,j,\Lambda}. \tag{2.10}$$

We know from (2.10) of [11] that

$$\omega_{G,N,j,\Lambda}^{(\alpha,\beta)}=\frac{\Gamma(N+\alpha+2)}{\beta^{\alpha}\Gamma(N+2)}\frac{1}{\xi_{G,N,j,\Lambda}^{(\alpha,\beta)}[\partial_x L_{N+1}^{(\alpha,\beta)}(\xi_{G,N,j,\Lambda}^{(\alpha,\beta)})]^2},\quad 0\le j\le N.$$

Next, let $m\ge 0$, and

$$\xi_{N,j,\Lambda}^{(m,\beta)}=\xi_{G,N-m,j,\Lambda}^{(m,\beta)},\qquad \omega_{N,j,\Lambda}^{(m,\beta)}=\omega_{G,N-m,j,\Lambda}^{(m,\beta)}(\xi_{G,N-m,j,\Lambda}^{(m,\beta)})^{-2m},\quad 0\le j\le N-m. \tag{2.11}$$

Thanks to (5.9) of [20], we have

$$\sum_{j=0}^{N-m}\phi(\xi_{N,j,\Lambda}^{(m,\beta)})\psi(\xi_{N,j,\Lambda}^{(m,\beta)})\omega_{N,j,\Lambda}^{(m,\beta)}=(\phi,\psi)_{\omega_{-m,\beta},\Lambda},\quad \forall\phi\in Q_N^{(-m,\beta)}(\Lambda),\,\psi\in Q_{N+1}^{(-m,\beta)}(\Lambda). \tag{2.12}$$

For any integer $r\ge 0$, we denote by $C^r(\Lambda)$ the space consisting of all r-times differentiable functions. Further, for integer $r\ge m-1$,

$$C_{0,m}^r(\Lambda)=\{v\in C^r(\Lambda)\mid \partial_x^k v(0)=0,\ 0\le k\le m-1\}.$$

For any $v\in C_{0,m}^{m-1}(\Lambda)$ and $m\ge 1$, the new Laguerre–Gauss–Radau type interpolation $\mathcal{I}_{N,-m,\beta,\Lambda}v\in Q_N^{(-m,\beta)}(\Lambda)$ is determined uniquely by

$$\mathcal{I}_{N,-m,\beta,\Lambda}v(\xi_{N,j,\Lambda}^{(m,\beta)})=v(\xi_{N,j,\Lambda}^{(m,\beta)}),\quad 0\le j\le N-m. \tag{2.13}$$

According to Lemma 5.2 of [20], we have the following result.

Lemma 2.3. *If $v\in {}_0H_{\omega_{-m,\beta},A}^m(\Lambda)\cap H_{\omega_{-m+k},\beta}^k(\Lambda)$, $\partial_x^r v\in L_{\omega_{-m+r,\beta}}^2(\Lambda)$ and integers $1\le m\le\min(r,N)$, then for $0\le k\le r\le N+1$,*

$$\|\partial_x^k(\mathcal{I}_{N,-m,\beta,\Lambda}v-v)\|_{\omega_{-m+k,\beta},\Lambda}\le c\left(\beta^{-\frac{1}{2}}+1\right)(\ln N)^{\frac{1}{2}}(\beta N)^{\frac{k+1-r}{2}}\|\partial_x^r v\|_{\omega_{-m+r,\beta},\Lambda}. \tag{2.14}$$

At the end of this section, we recall the following inequality (see Proposition 3.1 of [21]).

Lemma 2.4. *For any $v\in {}_0H_{\omega_{\alpha,\beta}}^1(\Lambda)$ and $\alpha\le 0$,*

$$\|v\|_{\omega_{\alpha,\beta},\Lambda}\le\frac{2}{\beta}\|\partial_x v\|_{\omega_{\alpha,\beta},\Lambda}. \tag{2.15}$$

3. New generalized Laguerre quasi-orthogonal approximation

In this section, we develop the new generalized Laguerre quasi-orthogonal approximation.

3.1. New generalized Laguerre orthogonal approximation

For any real numbers α,γ, and the numbers $\beta,\delta>0$, the new generalized Laguerre functions are defined by

$$\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x)=(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\mathcal{L}_l^{(\alpha,\beta)}(x),\quad l\ge\bar{l}_\alpha.$$

The special case with $\alpha=\gamma=0$ and $\beta=1$ was considered in [14]. Such basis functions were also used by Guo and Ma [16]. The case with $\alpha>-1$, $\gamma=0$ and $\beta>0$ was studied in [18]. The case with any real α, $\gamma=0$ and $\beta>0$ was investigated in [21]. We also refer the reader to the work of Everitt et al. [23] for the case with $\gamma=0$, $\beta=1$ and negative integer α, without the error estimate.

According to (2.2), $\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x)$ is the l-th eigenfunction of the following Sturm–Liouville problem,

$$\partial_x\left(x^{\alpha+1}e^{-\beta x}\partial_x\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right)\right)+\lambda_l^{(\alpha,\beta)}x^\alpha e^{-\frac{\beta}{2}x}(\delta+x)^{-\frac{\gamma}{2}}v(x)=0,\quad l\ge\bar{l}_\alpha. \tag{3.1}$$

Let $\hat{\omega}_{\alpha,\gamma,\delta}(x)=x^\alpha(\delta+x)^{-\gamma}$. By virtue of (2.4), the set of all $\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x)$ is a complete $L_{\hat{\omega}_{\alpha,\gamma,\delta}}^2(\Lambda)$-orthogonal system, and

$$\int_\Lambda\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x)\hat{\mathcal{L}}_{l'}^{(\alpha,\beta,\gamma,\delta)}(x)\hat{\omega}_{\alpha,\gamma,\delta}(x)dx=\eta_l^{(\alpha,\beta)}\delta_{l,l'}. \tag{3.2}$$

Thus, for any $v \in L^2_{\hat{\omega}_{\alpha,\gamma,\delta}}(\Lambda)$, we have

$$v(x) = \sum_{l=\bar{l}_\alpha}^{\infty} \hat{v}_l^{(\alpha,\beta,\gamma,\delta)} \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x) \tag{3.3}$$

with

$$\hat{v}_l^{(\alpha,\beta,\gamma,\delta)} = \frac{1}{\eta_l^{(\alpha,\beta)}} \int_\Lambda v(x) \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x) \hat{\omega}_{\alpha,\gamma,\delta}(x) dx. \tag{3.4}$$

Next, let

$$\hat{Q}_N^{(\alpha,\beta,\gamma,\delta)}(\Lambda) = \left\{ (\delta + x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x} \phi \mid \phi \in Q_N^{(\alpha,\beta)}(\Lambda) \right\}, \quad N \geq \bar{l}_\alpha.$$

The $L^2_{\hat{\omega}_{\alpha,\gamma,\delta}}(\Lambda)$-orthogonal projection $\hat{P}_{N,\alpha,\beta,\gamma,\Lambda} : L^2_{\hat{\omega}_{\alpha,\gamma,\delta}}(\Lambda) \to \hat{Q}_N^{(\alpha,\beta,\gamma,\delta)}(\Lambda)$ is defined by

$$(\hat{P}_{N,\alpha,\beta,\gamma,\delta,\Lambda} v - v, \phi)_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda} = 0, \quad \forall \phi \in \hat{Q}_N^{(\alpha,\beta,\gamma,\delta)}(\Lambda),$$

or equivalently,

$$\hat{P}_{N,\alpha,\beta,\gamma,\delta,\Lambda} v(x) = \sum_{l=\bar{l}_\alpha}^{N} \hat{v}_l^{(\alpha,\beta,\gamma,\delta)} \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x). \tag{3.5}$$

We now estimate the approximation error. To do this, we introduce the Sturm–Liouville operator:

$$\hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta} v(x) = -x^{-\alpha} e^{\frac{\beta}{2}x} (\delta + x)^{\frac{\gamma}{2}} \partial_x \left(x^{\alpha+1} e^{-\beta x} \partial_x \left((\delta + x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} v(x) \right) \right).$$

Thanks to (3.1), we have

$$\hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta} \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x) = \lambda_l^{(\alpha,\beta)} \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)}(x), \quad l \geq \bar{l}_\alpha. \tag{3.6}$$

By integration by parts, we derive that

$$\begin{aligned} (v, \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda} &= (\lambda_l^{(\alpha,\beta)})^{-1} (v, \hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta} \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda} \\ &= (\lambda_l^{(\alpha,\beta)})^{-1} (\hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta} v, \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}. \end{aligned} \tag{3.7}$$

Therefore, if u, v are in the domain of the operator $\hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta}$, then $(\hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta} u, v)_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda} = (u, \hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta} v)_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}$. Accordingly, $\hat{\mathcal{A}}_{\alpha,\beta,\gamma,\delta}$ is a positive definite and self-conjugate operator. Thus, we could define the following Sobolev-type spaces with integer $r \geq 0$,

$$D(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}) = \{ v \mid \hat{\mathcal{A}}^k_{\alpha,\beta,\gamma,\delta} v \in L^2_{\hat{\omega}_{\alpha,\gamma,\delta}}(\Lambda), 0 \leq k \leq r \},$$

equipped with the following semi-norm and norm,

$$|v|_{D(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta})} = \|\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta} v\|_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}, \qquad \|v\|_{D(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta})} = \left(\sum_{k=0}^{r} |v|^2_{D(\hat{\mathcal{A}}^k_{\alpha,\beta,\gamma,\delta})} \right)^{\frac{1}{2}}.$$

Theorem 3.1. *If $v \in D(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta})$ and integer $r \geq 0$, then for $0 \leq k \leq r \leq N+1$,*

$$|\hat{P}_{N,\alpha,\beta,\gamma,\delta,\Lambda} v - v|_{D(\hat{\mathcal{A}}^k_{\alpha,\beta,\gamma,\delta})} \leq c(\beta N)^{k-r} |v|_{D(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta})}. \tag{3.8}$$

Proof. We use (3.3), (3.5), (3.2) and (3.4) successively, to deduce that

$$\begin{aligned} |\hat{P}_{N,\alpha,\beta,\gamma,\delta,\Lambda} v - v|^2_{D(\hat{\mathcal{A}}^k_{\alpha,\beta,\gamma,\delta})} &= \sum_{l=N+1}^{\infty} (\lambda_l^{(\alpha,\beta)})^{2k} \eta_l^{(\alpha,\beta)} (\hat{v}_l^{(\alpha,\beta,\gamma,\delta)})^2 \\ &= \sum_{l=N+1}^{\infty} (\lambda_l^{(\alpha,\beta)})^{2k} (\eta_l^{(\alpha,\beta)})^{-1} (v, \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})^2_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}. \end{aligned}$$

Furthermore, by using (3.7) repeatedly, we obtain

$$(v, \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda} = (\lambda_l^{(\alpha,\beta)})^{-r} (\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta} v, \hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}.$$

The above two equalities, together with (2.3), lead to that

$$\begin{aligned}|\hat{P}_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v|^2_{D(\hat{\mathcal{A}}^k_{\alpha,\beta,\gamma,\delta})} &= \sum_{l=N+1}^{\infty}(\lambda_l^{(\alpha,\beta)})^{2k-2r}(\eta_l^{(\alpha,\beta)})^{-1}(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}v,\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})^2_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}\\ &\le c(\lambda_{N+1}^{(\alpha,\beta)})^{2k-2r}\sum_{l=N+1}^{\infty}(\eta_l^{(\alpha,\beta)})^{-1}(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}v,\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})^2_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}\\ &\le c(\beta N)^{2k-2r}\sum_{l=\bar{l}_\alpha}^{\infty}(\eta_l^{(\alpha,\beta)})^{-1}(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}v,\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma,\delta)})^2_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}.\end{aligned}$$

On the other hand, thanks to (3.2)–(3.4), we have

$$\|\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}v\|^2_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}=\sum_{l=\bar{l}_\alpha}^{\infty}(\eta_l^{(\alpha,\beta)})^{-1}(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}v,\hat{\mathcal{L}}_l^{(\alpha,\beta,\gamma)})^2_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}.$$

Consequently,

$$|\hat{P}_{N,\alpha,\beta,\gamma,\delta,\Lambda}v-v|_{D(\hat{\mathcal{A}}^k_{\alpha,\beta,\gamma,\delta})}\le c(\beta N)^{k-r}\|\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta}v\|_{\hat{\omega}_{\alpha,\gamma,\delta},\Lambda}=c(\beta N)^{k-r}|v|_{D(\hat{\mathcal{A}}^r_{\alpha,\beta,\gamma,\delta})}.$$

The proof is completed. □

3.2. Generalized Laguerre orthogonal approximation with negative integer α

We now focus on the special Laguerre orthogonal approximation with integer $\alpha < 0$, which is mostly appreciated for numerical solutions of high order differential equations defined on unbounded domains.

In the forthcoming discussions, we assume integer $m \ge 1$. Clearly,

$$\partial_x^k\hat{\mathcal{L}}_l^{(-m,\beta,\gamma,\delta)}(0)=0,\quad 0\le k\le m-1,\ l\ge m.$$

We introduce the following space with integer $r \ge 0$,

$$H^r_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)=\left\{v\mid v\text{ is measurable on }\Lambda\text{ and }\|v\|_{H^r_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)}<\infty\right\},$$

equipped with the semi-norm and the norm as

$$|v|_{H^r_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)}=\|\partial_x^r v\|_{\hat{\omega}_{-m+r,\gamma,\delta},\Lambda},\qquad \|v\|_{H^r_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)}=\left(\sum_{k=0}^{r}|v|^2_{H^k_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)}\right)^{\frac{1}{2}}.$$

Moreover, for $r \ge 1$,

$${}_0H^r_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)=\{v\mid v\in H^r_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)\text{ and }\partial_x^k v(0)=0\text{ for }0\le k\le r-1\}.$$

Theorem 3.2. *If $v\in {}_0H^m_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)\cap H^k_{\hat{\omega}_{-m+k,\gamma,\delta}}(\Lambda)$, $\partial_x^r((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v)\in L^2_{\omega_{-m+r,\beta}}(\Lambda)$, integers $1\le m\le\min(r,N)$ and $0\le k\le\min(r,m)$, then for $r\le N+1$,*

$$\|\partial_x^k(\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\le c(1+\beta^{-k})(\beta N)^{\frac{k-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.\tag{3.9}$$

Proof. Since $v\in {}_0H^m_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)\subset L^2_{\hat{\omega}_{-m,\gamma,\delta}}(\Lambda)$, we have

$$\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v(x)=\sum_{l=m}^{N}\hat{v}_l^{(-m,\beta,\gamma,\delta)}\hat{\mathcal{L}}_l^{(-m,\beta,\gamma,\delta)}(x)=(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\sum_{l=m}^{N}\hat{v}_l^{(-m,\beta,\gamma,\delta)}\mathcal{L}_l^{(-m,\beta)}(x).$$

With the aid of (3.4), it is easy to show that all coefficients $\hat{v}_l^{(-m,\beta,\gamma,\delta)}$ are exactly the same as the coefficients of expansion (2.7) for the function $(\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)$. Thus,

$$\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v(x)=(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right).$$

Accordingly, we use (2.8) with $k = 0$ to deduce that

$$\|\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v - v\|_{\hat{\omega}_{-m,\gamma,\delta,\Lambda}} = \|P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\|_{\omega_{-m,\beta},\Lambda}$$
$$\leq c(\beta N)^{-\frac{r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.$$

This is the result (3.9) with $k = 0$.

We next consider the case with $k \geq 1$. A careful calculation with $\delta \geq 1$ leads to that

$$\begin{aligned}
&\|\partial_x^k(\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v - v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\
&= \left\|\partial_x^k\left((\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right)\right\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\
&= \left\|\sum_{j=0}^{k}C_k^j\partial_x^{k-j}\left((\delta+x)^{\frac{\gamma}{2}}\right)\partial_x^j\left(e^{-\frac{\beta}{2}x}\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right)\right\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\
&\leq c\sum_{j=0}^{k}C_k^j\left\|\partial_x^j\left(e^{-\frac{\beta}{2}x}\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right)\right\|_{x^{-m+k},\Lambda}\\
&= c\sum_{j=0}^{k}C_k^j\left\|\sum_{i=0}^{j}C_j^i\partial_x^{j-i}\left(e^{-\frac{\beta}{2}x}\right)\partial_x^i\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{x^{-m+k},\Lambda}\\
&\leq c\sum_{j=0}^{k}\sum_{i=0}^{j}C_k^jC_j^i\left(\frac{\beta}{2}\right)^{j-i}\left\|\partial_x^i\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+k,\beta},\Lambda}.
\end{aligned} \tag{3.10}$$

Due to $v \in {}_0H^m_{\hat{\omega}_{-m,\gamma,\delta},\Lambda}(\Lambda)$ and $1 \leq k \leq m$, we have

$$\partial_x^i\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right)\right) = \partial_x^i\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right) = 0, \quad \text{for } x = 0,\ 0 \leq i \leq k-1.$$

Thus, by using (2.15) with $\alpha = -m + k \leq 0$ twice, we obtain from (3.10) that

$$\begin{aligned}
&\|\partial_x^k(\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v - v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\
&\leq c\sum_{j=0}^{k}\sum_{i=0}^{j}C_k^jC_j^i\left(\frac{\beta}{2}\right)^{j-i}\left(\frac{2}{\beta}\right)^{j-i}\left\|\partial_x^j\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+k,\beta},\Lambda}\\
&\leq c\sum_{j=0}^{k}C_k^j\left\|\partial_x^j\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+k,\beta},\Lambda}\\
&\leq c\sum_{j=0}^{k}C_k^j\left(\frac{2}{\beta}\right)^{k-j}\left\|\partial_x^k\left(P_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right) - (\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+k,\beta},\Lambda}.
\end{aligned}$$

Finally, we use (2.8) again to obtain

$$\|\partial_x^k(\hat{P}_{N,-m,\beta,\gamma,\delta,\Lambda}v - v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda} \leq c(1+\beta^{-k})(\beta N)^{\frac{k-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.$$

This ends the proof. □

3.3. Laguerre quasi-orthogonal approximation

For the spectral method of high order differential equations with mixed inhomogeneous boundary conditions, we have to consider other approximations. Moreover, in the multidomain spectral method, some derivatives of numerical solutions should keep the continuity on the common boundaries of adjacent subdomains. In this case, we also need certain unusual approximations. For this purpose, we introduce the Laguerre quasi-orthogonal approximation in this subsection.

Let $\theta_i = 1$ for $i = 3$, and $\theta_i = 0$ otherwise. Meanwhile, $\sigma_i = 1$ for $i \geq 2$, and $\sigma_i = 0$ otherwise. We also introduce the quantities

$$a_j(v) = \frac{1}{j!}\delta^{-\frac{\gamma}{2}}\sum_{i=0}^{j}(-1)^iC_j^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i + \frac{3}{4}\theta_i\gamma(\gamma\delta^{-1}-\beta)\delta^{-2} + \frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{j-i}v(0).$$

Furthermore, we set

$$v_{b,m,\gamma,\delta,\Lambda}(x) = (\delta+x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x} \sum_{j=0}^{m-1} a_j(v)x^j, \quad 1 \le m \le 4. \tag{3.11}$$

It can be checked that (see Appendix of this paper)

$$\partial_x^k v_{b,m,\gamma,\delta,\Lambda}(0) = \partial_x^k v(0), \quad 0 \le k \le m-1 \le 3. \tag{3.12}$$

For any $v \in H^m_{\hat\omega_{-m,\gamma,\delta},\Lambda}(\Lambda)$, we set $\hat v(x) = v(x) - v_{b,m,\gamma,\delta,\Lambda}(x)$. By virtue of (3.12), $\hat v \in {}_0H^m_{\hat\omega_{-m,\gamma,\delta},\Lambda}(\Lambda)$. Accordingly, we define the Laguerre quasi-orthogonal projection as

$$\hat P^m_{N,-m,\beta,\gamma,\delta,\Lambda} v(x) = (\delta+x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x} {}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v(x)\right) + v_{b,m,\gamma,\delta,\Lambda}(x). \tag{3.13}$$

Obviously,

$$\partial_x^k \hat P^m_{N,-m,\beta,\gamma,\delta,\Lambda} v(0) = \partial_x^k v(0), \quad 0 \le k \le m-1 \le 3. \tag{3.14}$$

Remark 3.1. The projection $\hat P^m_{N,-m,\beta,\gamma,\delta,\Lambda} v$ is similar to the orthogonal approximation ${}_0P^m_{N,-m,\beta,\Lambda} v$. But, it is not an orthogonal approximation. However, it keeps the same spectral accuracy as the orthogonal projection ${}_0P^m_{N,-m,\beta,\Lambda} v$; see the approximation result stated below. In particular, it fits the derivatives of order up to $m-1$ of the approximated functions exactly. Therefore, it is very useful for various multidomain spectral methods of high order problems.

Theorem 3.3. *If $v \in H^m_{\hat\omega_{-m,\gamma,\delta},\Lambda}(\Lambda) \cap H^k_{\hat\omega_{-m+k,\gamma,\delta}}(\Lambda)$, $\partial_x^r((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} v) \in L^2_{\omega_{-m+r,\beta}}(\Lambda)$, integers $1 \le m \le \min(4, r, N)$ and $0 \le k \le \min(r, m)$, then for $r \le N+1$,*

$$\|\partial_x^k(\hat P^m_{N,-m,\beta,\gamma,\delta,\Lambda} v - v)\|_{\hat\omega_{-m+k,\gamma,\delta},\Lambda} \le c(1+\beta^{-k})(\beta N)^{\frac{k-r}{2}} \left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}. \tag{3.15}$$

Proof. We have

$$\hat P^m_{N,-m,\beta,\gamma,\delta,\Lambda} v(x) - v(x) = (\delta+x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x}\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v(x)\right) - (\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v(x)\right).$$

Thanks to (2.9), it is easy to derive (3.15) with $k=0$. If $k \ge 1$, then

$$\partial_x^i\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v(x)\right)\right) = \partial_x^i\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v(x)\right) = 0, \quad \text{for } x=0,\ 0 \le i \le k-1. \tag{3.16}$$

We see from (3.11) that

$$\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} v_{b,m,\gamma,\delta,\Lambda}(x)\right) = 0, \quad \text{for } 1 \le m \le r. \tag{3.17}$$

Therefore, by using (2.15) with $\alpha = -m+k \le 0$ and (2.9), we verify that

$$\begin{aligned}
&\|\partial_x^k(\hat P^m_{N,-m,\beta,\gamma,\delta,\Lambda} v - v)\|_{\hat\omega_{-m+k,\gamma,\delta},\Lambda} \\
&= \left\|\partial_x^k\left((\delta+x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x}\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right) - (\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right)\right)\right\|_{\hat\omega_{-m+k,\gamma,\delta},\Lambda} \\
&= \left\|\sum_{j=0}^{k} C_k^j \partial_x^{k-j}\left((\delta+x)^{\frac{\gamma}{2}}\right)\partial_x^j\left(e^{-\frac{\beta}{2}x}\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right) - (\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right)\right)\right\|_{\hat\omega_{-m+k,\gamma,\delta},\Lambda} \\
&\le c\sum_{j=0}^{k} C_k^j \left\|\partial_x^j\left(e^{-\frac{\beta}{2}x}\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right) - (\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right)\right)\right\|_{x^{-m+k},\Lambda} \\
&= c\sum_{j=0}^{k} C_k^j \left\|\sum_{i=0}^{j} C_j^i \partial_x^{j-i}\left(e^{-\frac{\beta}{2}x}\right)\partial_x^i\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right) - (\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right)\right\|_{x^{-m+k},\Lambda} \\
&\le c\sum_{j=0}^{k} C_k^j \sum_{i=0}^{j} C_j^i \left(\frac{\beta}{2}\right)^{j-i}\left\|\partial_x^i\left({}_0P^m_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right) - (\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat v\right)\right\|_{\omega_{-m+k,\beta},\Lambda}.
\end{aligned} \tag{3.18}$$

Finally, following the same line as in the last part of proof of Theorem 3.2, we obtain from (3.18) and (3.17) that

$$\begin{aligned}\|\partial_x^k(\hat{P}^m_{N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda} &\le c(1+\beta^{-k})(\beta N)^{\frac{k-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{v}\right)\right\|_{\omega_{-m+r,\beta},\Lambda}\\ &= c(1+\beta^{-k})(\beta N)^{\frac{k-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.\end{aligned}$$

The proof is completed. □

4. New Laguerre–Gauss–Radau type interpolation

This section is devoted to the new Laguerre–Gauss–Radau type interpolation.

Let $\xi^{(m,\beta)}_{N,j,\Lambda}$ and $\omega^{(m,\beta)}_{N,j,\Lambda}$ be the same as in (2.11). We set

$$\begin{aligned}&\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}=\xi^{(m,\beta)}_{N,j,\Lambda},\quad 0\le j\le N-m,\ m\ge 0,\\ &\hat{\omega}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}=(\delta+\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})^{-\gamma}e^{\beta\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}}\omega^{(m,\beta)}_{N,j,\Lambda},\quad 0\le j\le N-m,\ m\ge 0.\end{aligned}\tag{4.1}$$

The discrete inner product and the norm are defined by

$$\begin{aligned}&(u,v)_{N,m,\beta,\gamma,\delta,\Lambda}=\sum_{j=0}^{N-m}u(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})v(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})\hat{\omega}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda},\\ &\|v\|_{N,m,\beta,\gamma,\delta,\Lambda}=(v,v)^{\frac12}_{N,m,\beta,\gamma,\delta,\Lambda}.\end{aligned}$$

For any $\phi\in\hat{Q}^{(-m,\beta,\gamma,\delta)}_N(\Lambda)$ and $\psi\in\hat{Q}^{(-m,\beta,\gamma,\delta)}_{N+1}(\Lambda)$, we have $\phi(x)=(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}q_\phi(x)$ and $\psi(x)=(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}q_\psi(x)$ where $q_\phi\in Q^{(-m,\beta)}_N(\Lambda)$ and $q_\psi\in Q^{(-m,\beta)}_{N+1}(\Lambda)$, respectively. Thereby, we use (2.12) to deduce that

$$\begin{aligned}(\phi,\psi)_{\hat{\omega}_{-m,\gamma,\delta},\Lambda}&=(q_\phi,q_\psi)_{\omega_{-m,\beta},\Lambda}=\sum_{j=0}^{N-m}q_\phi(\xi^{(m,\beta)}_{N,j,\Lambda})q_\psi(\xi^{(m,\beta)}_{N,j,\Lambda})\omega^{(m,\beta)}_{N,j,\Lambda}\\ &=\sum_{j=0}^{N-m}\phi(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})\psi(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})\hat{\omega}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}\\ &=(\phi,\psi)_{N,m,\beta,\gamma,\delta,\Lambda},\quad\forall\phi\in\hat{Q}^{(-m,\beta,\gamma,\delta)}_N(\Lambda),\ \psi\in\hat{Q}^{(-m,\beta,\gamma,\delta)}_{N+1}(\Lambda).\end{aligned}\tag{4.2}$$

In particular,

$$\|\phi\|_{\hat{\omega}_{-m,\gamma,\delta},\Lambda}=\|\phi\|_{N,m,\beta,\gamma,\delta,\Lambda},\quad\forall\,\phi\in\hat{Q}^{(-m,\beta,\gamma,\delta)}_N(\Lambda).\tag{4.3}$$

Let $C^r(\Lambda)$ and $C^r_{0,m}(\Lambda)$ be the same as in Section 2. For any $v\in C^{m-1}_{0,m}(\Lambda)$ and $m\ge 1$, the new auxiliary Laguerre–Gauss–Radau type interpolation $\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v\in\hat{Q}^{(-m,\beta,\gamma,\delta)}_N(\Lambda)$ is determined uniquely by

$$\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})=v(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}),\quad 0\le j\le N-m.\tag{4.4}$$

We next estimate the error of the interpolation $\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v$.

Theorem 4.1. *If $v\in{}_0H^m_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)\cap H^k_{\hat{\omega}_{-m+k,\gamma,\delta}}(\Lambda)$, $\partial_x^r((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v)\in L^2_{\omega_{-m+r,\beta}}(\Lambda)$, integers $1\le m\le\min(r,N)$ and $0\le k\le\min(r,m)$, then for $r\le N+1$,*

$$\begin{aligned}&\|\partial_x^k(\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\ &\quad\le c(1+\beta^{-k})\left(\beta^{-\frac12}+1\right)(\ln N)^{\frac12}(\beta N)^{\frac{k+1-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.\end{aligned}\tag{4.5}$$

Proof. We have from (2.13) and (4.4) that

$$\begin{aligned}&(\delta+\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}}\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})\\ &\quad=\mathcal{I}_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right)|_{x=\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}},\quad 0\le j\le N-m.\end{aligned}$$

Moreover, both of $(\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v(x)$ and $\mathcal{I}_{N,-m,\beta,\Lambda}((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x))$ belong to the same finite-dimensional set $Q_N^{(-m,\beta)}(\Lambda)$. This fact implies

$$\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v(x)=(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\mathcal{I}_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right).$$

Consequently,

$$\begin{aligned}&\|\partial_x^k(\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\&=\left\|\partial_x^k\left((\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\left(\mathcal{I}_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)-(\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right)\right\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}.\end{aligned}\tag{4.6}$$

A combination of (4.6) and (2.14) with $k=0$ leads to the result (4.5) with $k=0$.

We now consider the case with $k\ge 1$. Since $v\in {}_0H^m_{\hat{\omega}_{-m,\gamma,\delta},\Lambda}(\Lambda)$ and $\mathcal{I}_{N,-m,\beta,\Lambda}((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x))\in Q_N^{(-m,\beta)}(\Lambda)$, we assert that for $k\ge 1$,

$$\partial_x^i\left(\mathcal{I}_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right)-(\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v(x)\right)=0,\quad \text{for } x=0,\ 0\le i\le k-1.$$

Finally, by an argument similar to the proof of (3.15), together with (2.15) and (2.14), we derive from (4.6) that

$$\begin{aligned}&\|\partial_x^k(\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\&=\left\|\sum_{j=0}^k C_k^j\partial_x^{k-j}\left((\delta+x)^{\frac{\gamma}{2}}\right)\partial_x^j\left(e^{-\frac{\beta}{2}x}\left(\mathcal{I}_{N,-m,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)-(\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right)\right\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\&\le c(1+\beta^{-k})\left(\beta^{-\frac12}+1\right)(\ln N)^{\frac12}(\beta N)^{\frac{k+1-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.\end{aligned}$$

This ends the proof. □

We now turn to the new Laguerre–Gauss–Radau type interpolation for functions with inhomogeneous boundary values. Let $v_{b,m,\gamma,\delta,\Lambda}(x)$ be the same as in (3.11). For any $v\in C^{m-1}(\Lambda)$, we set $\hat{v}(x)=v(x)-v_{b,m,\gamma,\delta,\Lambda}(x)$. Clearly, $\hat{v}\in C^{m-1}_{0,m}(\Lambda)$. Thus, there exists the interpolation $\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}\hat{v}\in\hat{Q}_N^{(-m,\beta,\gamma,\delta)}(\Lambda)$. Then, we define the new interpolation by

$$\hat{\mathcal{I}}_{R,N,-m,\beta,\gamma,\delta,\Lambda}v(x)=\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}\hat{v}(x)+v_{b,m,\gamma,\delta,\Lambda}(x).$$

It can be checked that

$$\begin{aligned}&\hat{\mathcal{I}}_{R,N,-m,\beta,\gamma,\delta,\Lambda}v(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda})=v(\hat{\xi}^{(m,\beta,\gamma,\delta)}_{N,j,\Lambda}),\quad 0\le j\le N-m,\\&\partial_x^k\hat{\mathcal{I}}_{R,N,-m,\beta,\gamma,\delta,\Lambda}v(0)=\partial_x^k v(0),\quad 0\le k\le m-1.\end{aligned}\tag{4.7}$$

We now state the main result of this section.

Theorem 4.2. *If $v\in H^m_{\hat{\omega}_{-m,\gamma,\delta},A}(\Lambda)\cap H^k_{\hat{\omega}_{-m+k,\gamma,\delta}}(\Lambda)$, $\partial_x^r((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v)\in L^2_{\omega_{-m+r,\beta}}(\Lambda)$, integers $1\le m\le\min(4,r,N)$ and $0\le k\le\min(r,m)$, then for $r\le N+1$,*

$$\begin{aligned}&\|\partial_x^k(\hat{\mathcal{I}}_{R,N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\&\le c(1+\beta^{-k})\left(\beta^{-\frac12}+1\right)(\ln N)^{\frac12}(\beta N)^{\frac{k+1-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.\end{aligned}\tag{4.8}$$

Proof. Clearly,

$$\hat{\mathcal{I}}_{R,N,-m,\beta,\gamma,\delta,\Lambda}v(x)-v(x)=\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}\hat{v}(x)-\hat{v}(x).$$

This fact, along with (4.5) and (3.17), leads to that

$$\begin{aligned}&\|\partial_x^k(\hat{\mathcal{I}}_{R,N,-m,\beta,\gamma,\delta,\Lambda}v-v)\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}=\|\partial_x^k(\hat{\mathcal{I}}_{N,-m,\beta,\gamma,\delta,\Lambda}\hat{v}-\hat{v})\|_{\hat{\omega}_{-m+k,\gamma,\delta},\Lambda}\\&\le c(1+\beta^{-k})\left(\beta^{-\frac12}+1\right)(\ln N)^{\frac12}(\beta N)^{\frac{k+1-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{v}\right)\right\|_{\omega_{-m+r,\beta},\Lambda}\\&=c(1+\beta^{-k})\left(\beta^{-\frac12}+1\right)(\ln N)^{\frac12}(\beta N)^{\frac{k+1-r}{2}}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}v\right)\right\|_{\omega_{-m+r,\beta},\Lambda}.\end{aligned}$$

This completes the proof. □

5. Spectral and multidomain spectral methods

This section is for the spectral and multidomain spectral methods for high order problems defined on the half line.

5.1. Some preparations

Let $\lambda > 0$ and $\delta \geq 1$. We introduce the bilinear form

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u, v) = a_{1,\lambda,\delta,\Lambda}(u, v) + a_{2,\delta,\Lambda}(u, v), \quad \forall u, v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda), \tag{5.1}$$

with

$$a_{1,\lambda,\delta,\Lambda}(u, v) = \int_\Lambda \partial_x^2 u(x)\partial_x^2 v(x)(\delta + x)^{-\gamma} dx + \lambda \int_\Lambda u(x)v(x)(\delta + x)^{-\gamma} dx, \tag{5.2}$$

$$a_{2,\delta,\Lambda}(u, v) = -2\gamma \int_\Lambda \partial_x^2 u(x)\partial_x v(x)(\delta + x)^{-\gamma-1} dx + \gamma(\gamma + 1) \int_\Lambda \partial_x^2 u(x)v(x)(\delta + x)^{-\gamma-2} dx. \tag{5.3}$$

The space $H^1_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$ can be regarded as the interpolation between the spaces $H^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}(\Lambda)$ and $L^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}(\Lambda)$. Thus, there exists a positive constant $d_{\gamma,\delta}$ such that $\|\partial_x v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} \leq d_{\gamma,\delta}(\|\partial_x^2 v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \|v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})$. Thereby, a direct calculation shows that

$$\begin{aligned}
|\mathcal{A}_{\lambda,\delta,\Lambda}(u, v)| &\leq \frac{5}{4}\|\partial_x^2 u\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \frac{1}{2}\|\partial_x^2 v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + 4\gamma^2\|\partial_x v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} \\
&\quad + \frac{1}{2}\gamma^2(\gamma + 1)^2\|v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \frac{\lambda}{2}(\|u\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \|v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}) \\
&\leq \frac{5}{4}\|\partial_x^2 u\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \left(\frac{1}{2} + 4\gamma^2 d_{\gamma,\delta}\right)\|\partial_x^2 v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \frac{\lambda}{2}\|u\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} \\
&\quad + \frac{1}{2}(\lambda + 8\gamma^2 d_{\gamma,\delta} + \gamma^2(\gamma + 1)^2)\|v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \quad \forall u, v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda).
\end{aligned} \tag{5.4}$$

Next, for any $v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$, we have

$$\begin{aligned}
\gamma(\gamma + 1)\int_\Lambda \partial_x^2 v(x)v(x)(\delta + x)^{-\gamma-2} dx &= -\gamma(\gamma + 1)\int_\Lambda (\partial_x v(x))^2(\delta + x)^{-\gamma-2} dx + \gamma(\gamma + 1)(\gamma + 2) \\
&\quad \times \int_\Lambda \partial_x v(x)v(x)(\delta + x)^{-\gamma-3} dx - \gamma(\gamma + 1)\delta^{-\gamma-2}\partial_x v(0)v(0).
\end{aligned} \tag{5.5}$$

With the aid of (5.5), we use integration by parts to verify that

$$\begin{aligned}
&-2\gamma \int_\Lambda \partial_x^2 v(x)\partial_x v(x)(\delta + x)^{-\gamma-1} dx = -\gamma(\gamma + 1)\int_\Lambda (\partial_x v(x))^2(\delta + x)^{-\gamma-2} dx + \gamma\delta^{-\gamma-1}(\partial_x v(0))^2 \\
&= \gamma(\gamma + 1)\int_\Lambda \partial_x^2 v(x)v(x)(\delta + x)^{-\gamma-2} dx - \gamma(\gamma + 1)(\gamma + 2) \\
&\quad \times \int_\Lambda \partial_x v(x)v(x)(\delta + x)^{-\gamma-3} dx + \gamma(\gamma + 1)\delta^{-\gamma-2}\partial_x v(0)v(0) + \gamma\delta^{-\gamma-1}(\partial_x v(0))^2 \\
&= \gamma(\gamma + 1)\int_\Lambda \partial_x^2 v(x)v(x)(\delta + x)^{-\gamma-2} dx - \frac{1}{2}\gamma(\gamma + 1)(\gamma + 2)(\gamma + 3) \\
&\quad \times \int_\Lambda v^2(x)(\delta + x)^{-\gamma-4} dx + \gamma(\gamma + 1)\delta^{-\gamma-2}\partial_x v(0)v(0) + \gamma\delta^{-\gamma-1}(\partial_x v(0))^2 \\
&\quad + \frac{1}{2}\gamma(\gamma + 1)(\gamma + 2)\delta^{-\gamma-3}v^2(0).
\end{aligned} \tag{5.6}$$

For simplicity of the following discussions, we set

$$c_\gamma = 2\gamma^2(\gamma + 1)^2 + \frac{1}{2}\gamma(\gamma + 1)(\gamma + 2)(\gamma + 3).$$

Inserting (5.6) into (5.3) and using the Cauchy inequality, we derive that

$$\begin{aligned}
a_{2,\delta,\Lambda}(v, v) &= 2\gamma(\gamma + 1)\int_\Lambda \partial_x^2 v(x)v(x)(\delta + x)^{-\gamma-2} dx - \frac{1}{2}\gamma(\gamma + 1)(\gamma + 2)(\gamma + 3) \\
&\quad \times \int_\Lambda v^2(x)(\delta + x)^{-\gamma-4} dx + \gamma(\gamma + 1)\delta^{-\gamma-2}\partial_x v(0)v(0) + \gamma\delta^{-\gamma-1}(\partial_x v(0))^2 \\
&\quad + \frac{1}{2}\gamma(\gamma + 1)(\gamma + 2)\delta^{-\gamma-3}v^2(0)
\end{aligned}$$

$$\geq -\frac{1}{2}\int_\Lambda (\partial_x^2 v(x))^2(\delta+x)^{-\gamma}dx - \left(2\gamma^2(\gamma+1)^2+\frac{1}{2}\gamma(\gamma+1)(\gamma+2)(\gamma+3)\right)$$
$$\times \int_\Lambda v^2(x)(\delta+x)^{-\gamma-4}dx + \gamma(\gamma+1)\delta^{-\gamma-2}\partial_x v(0)v(0) + \gamma\delta^{-\gamma-1}(\partial_x v(0))^2$$
$$+\frac{1}{2}\gamma(\gamma+1)(\gamma+2)\delta^{-\gamma-3}v^2(0). \tag{5.7}$$

Then, by substituting (5.7) into (5.1) with $u = v$, we obtain that for any $v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$,

$$\mathcal{A}_{\lambda,\delta,\Lambda}(v,v) \geq \frac{1}{2}\|\partial_x^2 v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + (\lambda - c_\gamma)\|v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \gamma(\gamma+1)\delta^{-\gamma-2}\partial_x v(0)v(0) + \gamma\delta^{-\gamma-1}(\partial_x v(0))^2$$
$$+\frac{1}{2}\gamma(\gamma+1)(\gamma+2)\delta^{-\gamma-3}v^2(0). \tag{5.8}$$

On the other hand, if $\partial_x^3 u(x)v(x)(\delta+x)^{-\gamma} \to 0$ as $x \to \infty$, then by integrations by parts, we deduce that

$$\int_\Lambda \partial_x^4 u(x)v(x)(\delta+x)^{-\gamma}dx$$
$$= -\int_\Lambda \partial_x^3 u(x)\partial_x v(x)(\delta+x)^{-\gamma}dx + \gamma\int_\Lambda \partial_x^3 u(x)v(x)(\delta+x)^{-\gamma-1}dx - \delta^{-\gamma}\partial_x^3 u(0)v(0)$$
$$= \int_\Lambda \partial_x^2 u(x)\partial_x^2 v(x)(\delta+x)^{-\gamma}dx - 2\gamma\int_\Lambda \partial_x^2 u(x)\partial_x v(x)(\delta+x)^{-\gamma-1}dx + \gamma(\gamma+1)$$
$$\times \int_\Lambda \partial_x^2 u(x)v(x)(\delta+x)^{-\gamma-2}dx - \delta^{-\gamma}\partial_x^3 u(0)v(0) + \delta^{-\gamma}\partial_x^2 u(0)\partial_x v(0) - \gamma\delta^{-\gamma-1}\partial_x^2 u(0)v(0). \tag{5.9}$$

We now use an argument similar to the proof of Lemma 5.9 of [24], to derive an important property of the bilinear form $\mathcal{A}_{\lambda,\delta,\lambda}(u,v)$, which plays an important role in the spectral method. Let $W(\Lambda) \subseteqq H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$, $\overline{W}(\Lambda) \subseteqq H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda) \cap {}_0H^1_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$ and $\mathcal{Q}^*_N(\Lambda) \subset H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$ be a finite dimensional set. We set $W_N(\Lambda) = W(\Lambda) \cap \mathcal{Q}^*_N(\Lambda)$ and $\overline{W}_N(\Lambda) = \overline{W}(\Lambda) \cap \mathcal{Q}^*_N(\Lambda)$. We define the operator ${}_*P^2_{N,\lambda,\delta,\Lambda} : W(\Lambda) \to W_N(\Lambda)$, by

$$\mathcal{A}_{\lambda,\delta,\Lambda}({}_*P^2_{N,\lambda,\delta,\Lambda}v - v, \phi) = 0, \quad \forall\, \phi \in \overline{W}_N(\Lambda). \tag{5.10}$$

Proposition 5.1. *Let $v \in W(\Lambda)$, $w \in W_N(\Lambda)$, ${}_*P^2_{N,\lambda,\delta,\Lambda}v - w \in \overline{W}_N(\Lambda)$ and $\lambda > c_\gamma$. If, in addition, $\partial_x({}_*P^2_{N,\lambda,\delta,\Lambda}v - w)$ vanishes at $x = 0$ or $\gamma \geq 0$, then*

$$\mathcal{A}_{\lambda,\delta,\Lambda}(v - {}_*P^2_{N,\lambda,\delta,\Lambda}v, v - {}_*P^2_{N,\lambda}v) \leq \mathcal{A}_{\lambda,\delta,\Lambda}(v-w, v-w). \tag{5.11}$$

Proof. A direct calculation shows

$$\mathcal{A}_{\lambda,\delta,\Lambda}(v-w, v-w) = \mathcal{A}_{\lambda,\delta,\Lambda}(v - {}_*P^2_{N,\lambda,\delta,\Lambda}v, v - {}_*P^2_{N,\lambda,\delta,\Lambda}v)$$
$$+ \mathcal{A}_{\lambda,\delta,\Lambda}({}_*P^2_{N,\lambda,\delta,\Lambda}v - w, {}_*P^2_{N,\lambda,\delta,\Lambda}v - w) + 2\mathcal{A}_{\lambda,\delta,\Lambda}(v - {}_*P^2_{N,\lambda,\delta,\Lambda}v, {}_*P^2_{N,\lambda,\delta,\Lambda}v - w).$$

Thanks to (5.10), we have

$$\mathcal{A}_{\lambda,\delta,\Lambda}(v - {}_*P^2_{N,\lambda,\delta,\Lambda}v, {}_*P^2_{N,\lambda,\delta,\Lambda}v - w) = 0.$$

Due to $\lambda > c_\gamma$, (5.8) implies

$$\mathcal{A}_{\lambda,\delta,\Lambda}({}_*P^2_{N,\lambda,\delta,\Lambda}v - w, {}_*P^2_{N,\lambda,\delta,\Lambda}v - w) \geq 0.$$

Then, the desired result (5.11) follows from the previous statements immediately. □

5.2. The Dirichlet problem of the high order equation

We first consider the following simple model problem,

$$\begin{cases} \partial_x^4 U(x) + \lambda U(x) = f(x), & x \in \Lambda, \\ \partial_x U(0) = b, \quad U(0) = a, \end{cases} \tag{5.12}$$

where $f \in L^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$, a, b and $\lambda > 0$ are given constants. In addition, we require that $x^{\frac{1-\gamma}{2}}\partial_x^k U(x) \to 0$ as $x \to \infty$, for $0 \le k \le 2$, and $x^{\frac{-1-\gamma}{2}}\partial_x^3 U(x) \to 0$, as $x \to \infty$. Let

$$\tilde{H}^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda) = H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda) \cap \left\{ v \mid x^{\frac{-1-\gamma}{2}}\partial_x^3 v(x) \to 0,\ \text{as}\, x \to \infty \right\},$$
$$V(\Lambda) = \{v \mid v \in \tilde{H}^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda) \text{ and } \partial_x v(0) = b,\, v(0) = a\}, \qquad \overline{V}(\Lambda) = {}_0H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda).$$

Let $v \in \overline{V}(\Lambda)$. Multiplying (5.12) by $v(x)(\delta + x)^{-\gamma}$, and integrating the resulting equation by parts, we use (5.9) to derive a weak formulation of (5.12). It is to look for $U \in V(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(U, v) = (f, v)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \quad \forall\, v \in \overline{V}(\Lambda). \tag{5.13}$$

With the aid of (5.4) and (5.8), we show that if $\lambda > c_\gamma$, then the above problem admits a unique solution.

We now define the finite-dimensional spaces:

$$V_N(\Lambda) = V(\Lambda) \cap \hat{Q}_N^{(0,\beta,\gamma,\delta)}(\Lambda), \qquad \overline{V}_N(\Lambda) = \overline{V}(\Lambda) \cap \hat{Q}_N^{(0,\beta,\gamma,\delta)}(\Lambda).$$

The spectral method for (5.13), is to seek $u_N \in V_N(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N, \phi) = (f, \phi)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \quad \forall\, \phi \in \overline{V}_N(\Lambda). \tag{5.14}$$

For checking the existence of solutions of (5.14), it suffices to prove the uniqueness of the solution. Assume that $u_N^{(1)}(x)$ and $u_N^{(2)}(x)$ are solutions of (5.14), and $\tilde{u}_N(x) = u_N^{(1)}(x) - u_N^{(2)}(x) \in \overline{V}_N(\Lambda)$. Then

$$\mathcal{A}_{\lambda,\delta,\Lambda}(\tilde{u}_N, \phi) = 0, \quad \forall\, \phi \in \overline{V}_N(\Lambda).$$

Putting $\phi = \tilde{u}_N \in \overline{V}_N(\Lambda)$ in the above equation, we use (5.8) to obtain

$$\frac{1}{2}\|\partial_x^2 \tilde{u}_N\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + (\lambda - c_\gamma)\|\tilde{u}_N\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} \le \mathcal{A}_{\lambda,\delta,\Lambda}(\tilde{u}_N, \tilde{u}_N) = 0.$$

If $\lambda > c_\gamma$, then $\tilde{u}_N(x) \equiv 0$. This means the uniqueness of the solution of (5.14).

We now estimate the error of the numerical solution. For this purpose, we introduce the auxiliary operator $\overline{P}^2_{N,\lambda,\delta,\Lambda}$: $V(\Lambda) \to V_N(\Lambda)$, defined by

$$\mathcal{A}_{\lambda,\delta,\Lambda}(\overline{P}^2_{N,\lambda,\delta,\Lambda} v - v, \phi) = 0, \quad \forall\, \phi \in \overline{V}_N(\Lambda). \tag{5.15}$$

We have from (5.13) and (5.15) that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(\overline{P}^2_{N,\lambda,\delta,\Lambda} U, \phi) = (f, \phi)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.16}$$

Subtracting (5.16) from (5.14), yields

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U, \phi) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.17}$$

Taking $\phi = u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U \in \overline{V}_N(\Lambda)$ in (5.17), we obtain

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U, u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U) = 0.$$

The above equality and (5.8) imply $u_N = \overline{P}^2_{N,\lambda,\delta,\Lambda} U$.

So far, it remains to estimate the approximation error of the projection $\overline{P}^2_{N,\lambda,\delta,\Lambda} U$. For this purpose, we first estimate the error of the specific quasi-orthogonal projection $\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda} v(x)$. According to the definition (3.11) with $v = U$ and $m = 2$, we set

$$U_{b,2,\gamma,\delta,\Lambda}(x) = \delta^{-\frac{\gamma}{2}}(\delta + x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x}\left(U(0) + \left(\partial_x U(0) + \frac{\beta - \gamma\delta^{-1}}{2}U(0)\right)x\right),$$
$$\hat{U}(x) = U(x) - U_{b,2,\gamma,\delta,\Lambda}(x).$$

Then, by the definition (3.13), we have

$$\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda} U(x) = (\delta + x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x}\, {}_0P^2_{N,-2,\beta,\Lambda}\left((\delta + x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x}\hat{U}(x)\right) + U_{b,2,\gamma,\delta,\Lambda}(x). \tag{5.18}$$

A direct calculation shows that for integer $r \geq 2$,

$$
\begin{aligned}
&\|\partial_x^k(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}\\
&\quad=\left\|\partial_x^k\left((\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}-{}_0P^2_{N,-2,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}\right)\right)\right)\right\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}\\
&\quad=\left\|\sum_{j=0}^{k}C_k^j\partial_x^{k-j}\left((\delta+x)^{\frac{\gamma}{2}}\right)\partial_x^j\left(e^{-\frac{\beta}{2}x}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}-{}_0P^2_{N,-2,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}\right)\right)\right)\right\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}\\
&\quad\leq c\sum_{j=0}^{k}\left\|\partial_x^j\left(e^{-\frac{\beta}{2}x}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}-{}_0P^2_{N,-2,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}\right)\right)\right)\right\|^2_{\Lambda}. \qquad (5.19)
\end{aligned}
$$

On the other hand, it was shown in Proposition 2.1 of [21] that for any $v \in {}_0H^2_{\omega_{0,\beta}}(\Lambda)$,

$$\|\partial_x^2 v\|^2_{\omega_{0,\beta},\Lambda}=\left\|\partial_x^2\left(e^{-\frac{\beta}{2}x}v\right)\right\|^2_{\Lambda}+\frac{\beta^2}{2}\left\|\partial_x\left(e^{-\frac{\beta}{2}x}v\right)\right\|^2_{\Lambda}+\frac{\beta^4}{16}\|e^{-\frac{\beta}{2}x}v\|^2_{\Lambda},$$

whence

$$\left\|\partial_x^j\left(e^{-\frac{\beta}{2}x}v\right)\right\|^2_{\Lambda}\leq c\beta^{2j-4}\|\partial_x^2 v\|^2_{\omega_{0,\beta},\Lambda},\quad j=0,1,2. \qquad (5.20)$$

With the aid of (5.20) and (2.9) with $m=k=2$, we obtain from (5.19) that for $0\leq k\leq 2$,

$$
\begin{aligned}
\|\partial_x^k(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}&\leq c\sum_{j=0}^{k}\beta^{2j-4}\left\|\partial_x^2\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}-{}_0P^2_{N,-2,\beta,\Lambda}\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}\right)\right)\right\|^2_{\omega_{0,\beta},\Lambda}\\
&\leq c(1+\beta^{-4})(\beta N)^{2-r}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}\hat{U}\right)\right\|^2_{\omega_{-2+r,\beta},\Lambda}\\
&\leq c(1+\beta^{-4})(\beta N)^{2-r}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}U\right)\right\|^2_{\omega_{-2+r,\beta},\Lambda}. \qquad (5.21)
\end{aligned}
$$

Now, let $\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U$ and $\overline{P}^2_{N,\lambda,\delta,\Lambda}U$ be the same as in (5.18) and (5.15), respectively. We use Proposition 5.1 with

$$
\begin{aligned}
&W(\Lambda)=V(\Lambda),\qquad \overline{W}(\Lambda)=\overline{V}(\Lambda),\qquad W_N(\Lambda)=V_N(\Lambda),\qquad \overline{W}_N(\Lambda)=\overline{V}_N(\Lambda),\\
&v=U,\qquad w=\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U,\qquad {}_*P^2_{N,\lambda,\delta,\Lambda}U=\overline{P}^2_{N,\lambda,\delta,\Lambda}U.
\end{aligned}
$$

Then, by virtue of (5.8), (5.11) and (5.4), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\|\partial_x^2(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+(\lambda-c_\gamma)\|U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}\\
&\quad\leq \mathcal{A}_{\lambda,\delta,\Lambda}(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U,U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)\\
&\quad\leq \mathcal{A}_{\lambda,\delta,\Lambda}(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta\Lambda}U,U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\\
&\quad\leq\left(\frac{7}{4}+4\gamma^2d_{\gamma,\delta}\right)\|\partial_x^2(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}\\
&\qquad+(\lambda+8\gamma^2d_{\gamma,\delta}+\gamma^2(\gamma+1)^2)\|U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}. \qquad (5.22)
\end{aligned}
$$

Finally, we use the Gagliardo–Nirenberg inequality, (5.22) and (5.21) with $k=0,2$ successively, to derive that for $\lambda>c_\gamma$ and integer $r\geq 2$,

$$
\begin{aligned}
\|U-u_N\|^2_{H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)}&\leq c(\|\partial_x^2(U-u_N)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\lambda\|U-u_N\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})\\
&=c(\|\partial_x^2(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\lambda\|U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})\\
&\leq c(\|\partial_x^2(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\lambda\|U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})\\
&\leq c(1+\lambda)\left(1+\frac{1}{\beta^4}\right)(\beta N)^{2-r}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}U\right)\right\|^2_{\omega_{-2+r,\beta},\Lambda}. \qquad (5.23)
\end{aligned}
$$

Remark 5.1. In the case with $\lambda \le c_\gamma$, we could use the variable transformation $y = \theta x$ with $\theta > 0$. Accordingly, the problem (5.12) becomes

$$\begin{cases} \partial_y^4 V(y) + \dfrac{\lambda}{\theta^4} V(y) = \dfrac{1}{\theta^4} f(y), & y \in \Lambda, \\ \partial_x U(0) = \dfrac{b}{\theta}, & U(0) = a. \end{cases}$$

If θ is suitably small, then $\frac{\lambda}{\theta^4} > c_\gamma$.

5.3. The mixed boundary value problem of the high order equation

The Laguerre quasi-orthogonal approximation plays an important role in the spectral method for inhomogeneous boundary value problems. As an example, we consider the following model problem,

$$\begin{cases} \partial_x^4 U(x) + \lambda U(x) = f(x), & x \in \Lambda, \\ \partial_x^2 U(0) = b, & U(0) = a, \end{cases} \tag{5.24}$$

where $f \in L^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$, a, b and $\lambda > 0$ are given constants. In addition, we require that $x^{\frac{1-\gamma}{2}} \partial_x^k U(x) \to 0$ as $x \to \infty$ for $0 \le k \le 2$, and $x^{\frac{-1-\gamma}{2}} \partial_x^3 U(x) \to 0$, as $x \to \infty$. This problem is similar to the steady beam equation and the steady extended Fisher–Kolmogorov equation (cf. [25]). We define the following spaces,

$$V(\Lambda) = \{v \mid v \in \tilde{H}^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda) \text{ and } v(0) = a\}, \qquad \overline{V}(\Lambda) = \{v \mid v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda) \text{ and } v(0) = 0\},$$

and $\mathcal{A}_{\lambda,\delta,\Lambda}(u, v)$ be the same as in (5.1). We could use (5.9) to derive a weak formulation of (5.24). It is to find $U \in V(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(U, v) + b\delta^{-\gamma} \partial_x v(0) = (f, v)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \quad \forall v \in \overline{V}(\Lambda). \tag{5.25}$$

If $\lambda > c_\gamma$, then problem (5.25) admits a unique solution. Let

$$V_N(\Lambda) = V(\Lambda) \cap \hat{Q}_N^{(0,\beta,\gamma,\delta)}(\Lambda), \qquad \overline{V}_N(\Lambda) = \overline{V}(\Lambda) \cap \hat{Q}_N^{(0,\beta,\gamma,\delta)}(\Lambda).$$

The spectral method with $\gamma \ge 0$ for solving problem (5.25), is to seek $u_N \in V_N(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N, \phi) + b\delta^{-\gamma} \partial_x \phi(0) = (f, \phi)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.26}$$

For the existence of solutions of (5.26), it suffices to check the uniqueness of the solution. Assume that $u_N^{(1)}(x)$ and $u_N^{(2)}(x)$ are solutions of (5.26), and $\tilde{u}_N(x) = u_N^{(1)}(x) - u_N^{(2)}(x) \in \overline{V}_N(\Lambda)$. Then

$$\mathcal{A}_{\lambda,\delta,\Lambda}(\tilde{u}_N, \phi) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.27}$$

Putting $\phi = \tilde{u}_N$ in (5.27) and using (5.8) with $\gamma \ge 0$, we obtain

$$\frac{1}{2} \|\partial_x^2 \tilde{u}_N\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + (\lambda - c_\gamma) \|\tilde{u}_N\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \gamma \delta^{-\gamma-1} (\partial_x \tilde{u}_N(0))^2 \le \mathcal{A}_{\lambda,\delta,\Lambda}(\tilde{u}_N, \tilde{u}_N) = 0.$$

If $\lambda > c_\gamma$, then the above inequality implies $\tilde{u}_N(x) \equiv 0$. This means the existence and the uniqueness of the solution of (5.26).

We now turn to deal with the convergence of (5.26). We introduce the operator $\overline{P}^2_{N,\lambda,\delta,\Lambda} : V(\Lambda) \to V_N(\Lambda)$, defined by

$$\mathcal{A}_{\lambda,\delta,\Lambda}(\overline{P}^2_{N,\lambda,\delta,\Lambda} v - v, \phi) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.28}$$

We have from (5.25) and (5.28) that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(\overline{P}^2_{N,\lambda,\delta,\Lambda} U, \phi) + b\delta^{-\gamma} \partial_x \phi(0) = (f, \phi)_{\hat{\omega}_{0,\gamma,\delta}}, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.29}$$

Subtracting (5.29) from (5.26), yields

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U, \phi) = 0, \quad \forall \phi \in \overline{V}_N(\Lambda). \tag{5.30}$$

Taking $\phi = u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U \in \overline{V}_N(\Lambda)$ in (5.30), we obtain

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U, u_N - \overline{P}^2_{N,\lambda,\delta,\Lambda} U) = 0.$$

Using (5.8) with $\gamma \ge 0$ again, we assert that $u_N = \overline{P}^2_{N,\lambda,\delta,\Lambda} U$.

We now use Proposition 5.1 with

$$W(\Lambda)=V(\Lambda),\qquad \overline{W}(\Lambda)=\overline{V}(\Lambda),\qquad W_N(\Lambda)=V_N(\Lambda),\qquad \overline{W}_N(\Lambda)=\overline{V}_N(\Lambda),$$
$$v=U,\qquad w=\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U,\qquad {}_*P^2_{N,\lambda,\delta,\Lambda}U=\overline{P}^2_{N,\lambda,\delta,\Lambda}U.$$

Since $\gamma\geq 0$, we could use (5.8), (5.11) and (5.14) to verify that

$$\begin{aligned}
&\frac{1}{2}\|\partial_x^2(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+(\lambda-c_\gamma)\|U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\gamma\delta^{-\gamma-1}(\partial_x(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)(0))^2\\
&\quad\leq \mathcal{A}_{\lambda,\delta,\Lambda}(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U,U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)\\
&\quad\leq \mathcal{A}_{\lambda,\delta,\Lambda}(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U,U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\\
&\quad\leq \left(\frac{7}{4}+4\gamma^2 d_{\gamma,\delta}\right)\|\partial_x^2(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}\\
&\quad\quad+(\lambda+8\gamma^2 d_{\gamma,\delta}+\gamma^2(\gamma+1)^2)\|U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}.
\end{aligned}$$

Finally, we use the Gagliardo–Nirenberg inequality and (5.21) to verify that for $\lambda>c_\gamma$, $\gamma\geq 0$ and integer $r\geq 2$,

$$\begin{aligned}
\|U-u_N\|^2_{H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)}&\leq c(\|\partial_x^2(U-u_N)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\lambda\|U-u_N\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})\\
&=c(\|\partial_x^2(U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\lambda\|U-\overline{P}^2_{N,\lambda,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})\\
&\leq c(\|\partial_x^2(U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}+\lambda\|U-\hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda}U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda})\\
&\leq c(1+\lambda)\left(1+\frac{1}{\beta^4}\right)(\beta N)^{2-r}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}U\right)\right\|^2_{\omega_{-2+r,\beta},\Lambda}.
\end{aligned}\tag{5.31}$$

Remark 5.2. We may design the spectral method with essential imposition of mixed boundary conditions. Such trick provides better numerical results sometimes; see [26,21]. To show this, we set

$$V^*_N(\Lambda)=V(\Lambda)\cap\hat{Q}_N^{(0,\beta,\gamma,\delta)}(\Lambda)\cap\{v\mid\partial_x^2 v(0)=b\},$$
$$\overline{V}^*_N(\Lambda)=\overline{V}(\Lambda)\cap\hat{Q}_N^{(0,\beta,\gamma,\delta)}(\Lambda)\cap\{v\mid\partial_x^2 v(0)=0\}.$$

The spectral method with $\gamma\geq 0$ for solving problem (5.25), is to find $u_N\in V^*_N(\Lambda)$ such that

$$\mathcal{A}_{\lambda,\delta,\Lambda}(u_N,\phi)+b\delta^{-\gamma}\partial_x\phi(0)=(f,\phi)_{\hat{\omega}_{0,\gamma,\delta},\Lambda},\quad\forall\phi\in\overline{V}^*_N(\Lambda).\tag{5.32}$$

Following the same line as in the derivation of (5.31), we could use Proposition 5.1 with $W(\Lambda)=V(\Lambda)$, $\overline{W}(\Lambda)=\overline{V}(\Lambda)$, $W_N(\Lambda)=V^*_N(\Lambda)$, $\overline{W}_N(\Lambda)=\overline{V}^*_N(\Lambda)$, $v=U$, $w=\hat{P}^3_{N,-3,\beta,\gamma,\delta,\Lambda}U$, and ${}_*P^2_{N,\lambda,\delta,\Lambda}U=\overline{P}^2_{N,\lambda,\delta,\Lambda}U\in V^*_N(\Lambda)$, to reach that if $\lambda>c_\gamma$ and integer $r\geq 3$, then

$$\|U-u_N\|^2_{H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)}\leq\frac{c}{\beta^2}(1+\lambda)\left(1+\frac{1}{\beta^4}\right)(\beta N)^{3-r}\left\|\partial_x^r\left((\delta+x)^{-\frac{\gamma}{2}}e^{\frac{\beta}{2}x}U\right)\right\|^2_{\omega_{-3+r,\beta},\Lambda}.\tag{5.33}$$

The scheme (5.32) provides better numerical results. Whereas, unfortunately, the error estimate (5.33) is not as good as the error estimate (5.31) for the scheme (5.26) without essential imposition of mixed boundary conditions. How to improve the error estimate (5.33) is still an open problem.

5.4. The multidomain spectral method

In many practical problems, the solutions vary rapidly near the fixed boundaries. For raising the numerical accuracy and saving work, we might use the Legendre approximation near the boundaries, coupled with the Laguerre approximation on the remaining domains; cf. [16,27]. To do this, we need some additional preparations.

Let $I=(-2,0)$, $m,n\geq 1$, and

$$\chi^{(m,n)}(x)=(-x)^m(2+x)^n,\quad x\in I.$$

We denote the inner product and the norm of the weighted space $L^2_{\chi^{(m,n)}}(I)$ by $(\cdot,\cdot)_{\chi^{(m,n)},I}$ and $\|\cdot\|_{\chi^{(m,n)},I}$, respectively. For integer $r\geq 0$, we introduce the space

$$H^r_{m,n,A}(I)=\{v\mid v\text{ is measurable on }I\text{ and }\|v\|_{H^r_{m,n,A}(I)}<\infty\},$$

equipped with the following semi-norm and norm,

$$|v|_{H^r_{m,n,A}(I)} = \|\partial_x^r v\|_{\chi^{(-m+r,-n+r)},I}, \qquad \|v\|_{H^r_{m,n,A}(I)} = \left(\sum_{k=0}^{r} |v|^2_{H^k_{m,n,A}(I)}\right)^{\frac{1}{2}}.$$

Moreover, for integer $r > \max(m, n)$,

$$H^r_{0,m,n,A}(I) = \{v \in H^r_{m,n,A}(I) \mid \partial_x^k v(-2) = 0 \text{ for } 0 \le k \le n-1, \text{ and } \partial_x^k v(0) = 0 \text{ for } 0 \le k \le m-1\}.$$

Next, let $J_l^{(m,n)}(x)$ be the standard Jacobi polynomial of degree l, and

$$Y_l^{(m,n)}(x) = \chi^{(m,n)}(x) J_{l-m-n}^{(m,n)}(x+1), \quad x \in I, l \ge m+n,$$
$$Q_N^{(m,n)}(I) = \operatorname{span}\{Y_l^{(m,n)}(x),\ m+n \le l \le N\}.$$

For integers $\max(m, n) \le \mu \le m + n$, the projection $P^{\mu,0}_{N,m,n,I} : H^\mu_{0,m,n,A}(I) \to Q_N^{(m,n)}(I)$ is defined by

$$(\partial_x^\mu (v - P^{\mu,0}_{N,m,n,I} v), \partial_x^\mu \phi)_{\chi^{(-m+\mu,-n+\mu)}} = 0, \quad \forall \phi \in Q_N^{(m,n)}(I).$$

We next introduce the polynomials $q^-_{m,n,j,I}(x), q^+_{m,n,j,I}(x) \in \mathcal{P}_{m+n-1}(I)$ as follows,

$$q^-_{m,n,j,I}(x) = \frac{1}{2^m j!}(-x)^m \sum_{l=0}^{n-1-j} \frac{(m+l-1)!}{2^l l!(m-1)!}(2+x)^{l+j},$$
$$q^+_{m,n,j,I}(x) = \frac{(-1)^j}{2^n j!}(2+x)^n \sum_{l=0}^{m-1-j} \frac{(n+l-1)!}{2^l l!(n-1)!}(-x)^{l+j}. \tag{5.34}$$

It can be checked that

$$\partial_x^k q^-_{m,n,j,I}(-2) = \delta_{k,j}, \qquad \partial_x^l q^-_{m,n,j,I}(0) = 0, \quad 0 \le j, k \le n-1,\ 0 \le l \le m-1,$$
$$\partial_x^l q^+_{m,n,j,I}(-2) = 0, \qquad \partial_x^k q^+_{m,n,j,I}(0) = \delta_{k,j}, \quad 0 \le j, k \le m-1,\ 0 \le l \le n-1. \tag{5.35}$$

For any $v \in H^r_{m,n,A}(I)$ and $r \ge \max(m, n)$, we set

$$v_{m,n,b,I}(x) = \sum_{j=0}^{n-1} \partial_x^j v(-2) q^-_{m,n,j,I}(x) + \sum_{j=0}^{m-1} \partial_x^j v(0) q^+_{m,n,j,I}(x).$$

Let $\tilde{v}(x) = v(x) - v_{m,n,b,I}(x)$. The Jacobi quasi-orthogonal projection was defined by (cf. (2.18) of [20])

$$P^\mu_{N,m,n,I} v(x) = P^{\mu,0}_{N,m,n,I} \tilde{v}(x) + v_{m,n,b,I}(x). \tag{5.36}$$

According to the results (2.21) and (2.24) of [20], we know that if $v \in H^\mu_{m,n,A}(I)$, $\partial_x^r v \in L^2_{\chi^{(-m+r,-n+r)}}(I)$, integers $m, n, r \ge 1$, $N \ge m+n$, $0 \le k \le r \le N+1$, $\max(m, n, k) \le \mu \le m+n$, and $r \ge m+n$(or $\max(m, n) \le r \le m+n-1$, $1 \le m, n \le 4$), then

$$\|\partial_x^k (P^\mu_{N,m,n,I} v - v)\|_{\chi^{(-m+k,-n+k)},I} \le cN^{k-r} \|\partial_x^r v\|_{\chi^{(-m+r,-n+r)},I}. \tag{5.37}$$

Now, let $\Omega = I \cup \Lambda = \{x \mid -2 < x < \infty\}$. We denote the inner product and norm of $L^2_\chi(\Omega)$ by $(\cdot,\cdot)_\chi$ and $\|\cdot\|_\chi$, respectively. We omit χ in notations, whenever $\chi \equiv 1$. Let $\delta > 2$, and $d_{\gamma,\delta}$ be the positive constant such that $\|\partial_x v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} \le d_{\gamma,\delta}(\|\partial_x^2 v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} + \|v\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega})$.

In this subjection, we always assume $\delta \ge 2$. We introduce the bilinear form

$$\mathcal{A}_{\lambda,\gamma,\delta,\Omega}(u, v) = a_{1,\lambda,\gamma,\delta,\Omega}(u, v) + a_{2,\gamma,\delta,\Omega}(u, v), \quad \forall u, v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Omega), \tag{5.38}$$

with

$$a_{1,\lambda,\gamma,\delta,\Omega}(u, v) = \int_\Omega \partial_x^2 u(x) \partial_x^2 v(x)(\delta+x)^{-\gamma} dx + \lambda \int_\Omega u(x)v(x)(\delta+x)^{-\gamma} dx,$$
$$a_{2,\gamma,\delta,\Omega}(u, v) = -2\gamma \int_\Omega \partial_x^2 u(x) \partial_x v(x)(\delta+x)^{-\gamma-1} dx + \gamma(\gamma+1) \int_\Omega \partial_x^2 u(x) v(x)(\delta+x)^{-\gamma-2} dx.$$

By an argument similar to the derivation of (5.4), we obtain

$$|\mathcal{A}_{\lambda,\gamma,\delta,\Omega}(u,v)| \le \frac{5}{4}\|\partial_x^2 u\|^2_{\hat\omega_{0,\gamma,\delta},\Omega} + \left(\frac{1}{2}+4\gamma^2 d_{\gamma,\delta}\right)\|\partial_x^2 v\|^2_{\hat\omega_{0,\gamma,\delta},\Omega} + \frac{\lambda}{2}\|u\|^2_{\hat\omega_{0,\gamma,\delta},\Omega} + \frac{1}{2}(\lambda+8\gamma^2 d_{\gamma,\delta}+\gamma^2(\gamma+1)^2)\|v\|^2_{\hat\omega_{0,\gamma,\delta},\Omega}, \quad \forall u,v\in H^2_{\hat\omega_{0,\gamma,\delta},\Omega}(\Omega). \tag{5.39}$$

Also, following the same line as in the derivation of (5.8), we derive that for any $v\in H^2_{\hat\omega_{0,\gamma,\delta}}(\Omega)$,

$$\mathcal{A}_{\lambda,\gamma,\delta,\Omega}(v,v) \ge \frac{1}{2}\|\partial_x^2 v\|^2_{\hat\omega_{0,\gamma,\delta},\Omega} + (\lambda-c_\gamma)\|v\|^2_{\hat\omega_{0,\gamma,\delta},\Omega} + \gamma(\gamma+1)(\delta-2)^{-\gamma-2}\partial_x v(-2)v(-2) + \gamma(\delta-2)^{-\gamma-1}(\partial_x v(-2))^2 + \frac{1}{2}\gamma(\gamma+1)(\gamma+2)(\delta-2)^{-\gamma-3}v^2(-2). \tag{5.40}$$

On the other hand, if $\partial_x^3 u(x)v(x)(\delta+x)^{-\gamma}\to 0$ as $x\to\infty$, then

$$\int_\Omega \partial_x^4 u(x)v(x)(\delta+x)^{-\gamma}dx = \int_\Omega \partial_x^2 u(x)\partial_x^2 v(x)(\delta+x)^{-\gamma}dx - 2\gamma\int_\Omega \partial_x^2 u(x)\partial_x v(x)(\delta+x)^{-\gamma-1}dx + \gamma(\gamma+1)\int_\Omega \partial_x^2 u(x)v(x)(\delta+x)^{-\gamma-2}dx - (\delta-2)^{-\gamma}\partial_x^3 u(-2)v(-2) + (\delta-2)^{-\gamma}\partial_x^2 u(-2)\partial_x v(-2) - \gamma(\delta-2)^{-\gamma-1}\partial_x^2 u(-2)v(-2). \tag{5.41}$$

Next, let $W(\Omega), \overline{W}(\Omega) \subseteq H^2_{\hat\omega_{0,\gamma,\delta}}(\Omega)$. For the pair $\mathbf{N}=(N_1,N_2)$, we set

$$W_{\mathbf{N}}(\Omega) = \{\phi\in W(\Omega) \mid \phi|_I\in P_{N_1}(I), \phi|_\Lambda \in \hat Q_{N_2}^{(0,\beta,\gamma,\delta)}(\Lambda)\},$$
$$\overline{W}_{\mathbf{N}}(\Omega) = \{\phi\in \overline{W}(\Omega) \mid \phi|_I\in P_{N_1}(I), \phi|_\Lambda \in \hat Q_{N_2}^{(0,\beta,\gamma,\delta)}(\Lambda)\}.$$

We define the operator ${}_*P^2_{\mathbf{N},\lambda,\delta,\Omega} : W(\Omega)\to W_{\mathbf{N}}(\Omega)$, by

$$\mathcal{A}_{\lambda,\delta,\Omega}({}_*P^2_{\mathbf{N},\lambda,\delta,\Omega}v - v, \phi) = 0, \quad \forall\phi\in\overline{W}_{\mathbf{N}}(\Omega). \tag{5.42}$$

With the aid of (5.40), we can prove the following property in the same manner as for the proof of Proposition 5.1.

Proposition 5.2. *Let $v\in W(\Omega)$, $w\in W_{\mathbf{N}}(\Omega)$, ${}_*P^2_{\mathbf{N},\lambda,\delta,\Omega}v - w\in\overline{W}_{\mathbf{N}}(\Omega)$, $\lambda > c_\gamma$ and $\delta\ge 2$. If $\partial_x({}_*P^2_{\mathbf{N},\lambda,\delta,\Omega}v - w) = 0$ at $x=-2$, or $\gamma\ge 0$, or $\delta = 2$, then*

$$\mathcal{A}_{\lambda,\delta,\Omega}(v - {}_*P^2_{\mathbf{N},\lambda,\delta,\Omega}v, v - {}_*P^2_{\mathbf{N},\lambda,\Omega}v) \le \mathcal{A}_{\lambda,\delta,\Omega}(v-w, v-w). \tag{5.43}$$

We now consider the following problem,

$$\begin{cases}\partial_x^4 U(x) + \lambda U(x) = f(x), & x\in\Omega,\\ \partial_x U(-2) = b, \quad U(-2) = a,\end{cases} \tag{5.44}$$

where $f\in L^2_{\hat\omega_{0,\gamma,\delta}}(\Omega)$, a, b and $\lambda > 0$ are given constants. In addition, we require that $x^{\frac{1-\gamma}{2}}\partial_x^k U(x)\to 0$, as $x\to\infty$, $0\le k\le 2$, and $x^{\frac{-1-\gamma}{2}}\partial_x^3 U(x)\to 0$, as $x\to\infty$.

Let

$$\tilde H^2_{\hat\omega_{0,\gamma,\delta}}(\Omega) = H^2_{\hat\omega_{0,\gamma,\delta}}(\Omega)\cap\left\{v \mid x^{\frac{-1-\gamma}{2}}\partial_x^3 v(x)\to 0, \text{ as } x\to\infty\right\},$$
$$V(\Omega) = \{v \mid v\in\tilde H^2_{\hat\omega_{0,\gamma,\delta}}(\Omega) \text{ and } \partial_x v(-2) = b, v(-2) = a\},$$
$$\overline{V}(\Omega) = \{v \mid v\in H^2_{\hat\omega_{0,\gamma,\delta}}(\Omega) \text{ and } \partial_x v(-2) = v(-2) = 0\}.$$

We use (5.41) to derive a weak form of (5.44). It is to look for $U\in V(\Omega)$ such that

$$\mathcal{A}_{\lambda,\delta,\Omega}(U,v) = (f,v)_{\hat\omega_{0,\gamma,\delta},\Omega}, \quad \forall v\in\overline{V}(\Omega). \tag{5.45}$$

If $\lambda > c_\gamma$, then the above problem admits a unique solution.

Next, let

$$V_{\mathbf{N}}(\Omega) = \{\phi\in V(\Omega) \mid \phi|_I\in P_{N_1}(I), \phi|_\Lambda\in\hat Q_{N_2}^{(0,\beta,\gamma,\delta)}(\Lambda)\},$$
$$\overline{V}_{\mathbf{N}}(\Omega) = \{\phi\in \overline{V}(\Omega) \mid \phi|_I\in P_{N_1}(I), \phi|_\Lambda\in\hat Q_{N_2}^{(0,\beta,\gamma,\delta)}(\Lambda)\}.$$

The multidomain spectral method for solving problem (5.45), is to seek $u_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega)$ such that

$$\mathcal{A}_{\lambda,\delta,\Omega}(u_{\mathbf{N}}, \phi) = (f, \phi)_{\hat{\omega}_{0,\gamma,\delta,\Omega}}, \quad \forall \phi \in \overline{V}_{\mathbf{N}}(\Omega). \tag{5.46}$$

If $\lambda > c_\gamma$, and $\gamma \ge 0$ or $\delta = 2$, then the problem (5.46) has a unique solution.

We now estimate the error of the numerical solution. For this purpose, we introduce the composite quasi-orthogonal projection $P^2_{\mathbf{N},\gamma,\delta,\Omega} v(x)$, defined by

$$P^2_{\mathbf{N},\gamma,\delta,\Omega} v(x) = \begin{cases} P^2_{N_1,2,2,I} v(x), & x \in I, \\ \hat{P}^2_{N_2,-2,\beta,\gamma,\delta,\Lambda} v(x), & x \in \Lambda. \end{cases} \tag{5.47}$$

Obviously, $P^2_{\mathbf{N},\gamma,\delta,\Omega} U \in V_{\mathbf{N}}(\Omega)$.

Moreover, if $\delta > 2$, or $\delta = 2$ and $\gamma \le 0$, then

$$\|\partial_x^k v\|^2_{\hat{\omega}_{0,\gamma,\delta},I} \le c\|\partial_x^k v\|^2_{\chi^{(-2+k,2+k)},I}, \quad 0 \le k \le 2. \tag{5.48}$$

Therefore, we use (5.37) to deduce that for integer $r_1 \ge 2$,

$$\begin{aligned} \|\partial_x^k (U - P^2_{N_1,2,2,I} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},I} &\le c\|\partial_x^k (U - P^2_{N_1,2,2,I} U)\|^2_{\chi^{(-2+k,-2+k)},I} \\ &\le c N_1^{2k-2r_1} \|\partial_x^{r_1} U\|^2_{\chi^{(-2+r_1,-2+r_1)},I}, \quad k = 0, 1, 2. \end{aligned} \tag{5.49}$$

On the other hand, (5.21) implies that for integer $r_2 \ge 2$,

$$\|\partial_x^k (U - \hat{P}^2_{N,-2,\beta,\gamma,\delta,\Lambda} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} \le c(1+\beta^{-4})(\beta N)^{2-r_2} \left\| \partial_x^{r_2} \left((\delta + x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} U \right) \right\|^2_{\omega_{-2+r_2,\beta},\Lambda}, \quad k = 0, 1, 2. \tag{5.50}$$

Furthermore, we introduce the auxiliary operator $\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} : V(\Omega) \to V_{\mathbf{N}}(\Omega)$, defined by

$$\mathcal{A}_{\lambda,\delta,\Omega}(\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} v - v, \phi) = 0, \quad \forall \phi \in \overline{V}_{\mathbf{N}}(\Omega). \tag{5.51}$$

It can be shown as before that $u_{\mathbf{N}} = \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$. Thus, it suffices to estimate the approximation error of $\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$.

Now, let $P_{\mathbf{N},\gamma,\delta,\Omega} U$ and $\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$ be the same as in (5.47) and (5.51), respectively. We use Proposition 5.2, with $W(\Omega) = V(\Omega)$, $\overline{W}(\Omega) = \overline{V}(\Omega)$, $W_{\mathbf{N}}(\Omega) = V_{\mathbf{N}}(\Omega)$, $\overline{W}_{\mathbf{N}}(\Omega) = \overline{V}_{\mathbf{N}}(\Omega)$, $v = U$, $w = P^2_{\mathbf{N},\gamma,\delta,\Omega} U$ and ${}_*P^2_{\mathbf{N},\lambda,\delta,\Omega} U = \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$. Consequently, by virtue of (5.40), (5.43) and (5.39), we derive that

$$\begin{aligned} &\frac{1}{2}\|\partial_x^2 (U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} + (\lambda - c_\gamma)\|U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} \\ &\quad \le \mathcal{A}_{\lambda,\delta,\Omega}(U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U, U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U) \\ &\quad \le \mathcal{A}_{\lambda,\delta,\Omega}(U - P^2_{\mathbf{N},\gamma,\delta,\Omega} U, U - P^2_{\mathbf{N},\gamma,\delta,\Omega} U) \\ &\quad \le \left(\frac{7}{4} + 4\gamma^2 d_{\gamma,\delta} \right) \|\partial_x^2 (U - P^2_{\mathbf{N},\gamma,\delta,\Omega} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + (\lambda + 8\gamma^2 d_{\gamma,\delta} + \gamma^2(\gamma+1)^2)\|U - P^2_{\mathbf{N},\gamma,\delta,\Omega} U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}. \end{aligned} \tag{5.52}$$

Finally, we use the Gagliardo–Nirenberg inequality, (5.52), (5.49), (5.50) successively, to obtain that if $\lambda > c_\gamma$, and $\delta > 2$ (or $\gamma \le 0$ and $\delta = 2$), and integers $r_1, r_2 \ge 2$,

$$\begin{aligned} \|U - u_{\mathbf{N}}\|^2_{H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Omega)} &\le c(\|\partial_x^2 (U - u_{\mathbf{N}})\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \lambda\|U - u_{\mathbf{N}}\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega}) \\ &= c(\|\partial_x^2 (U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} + \lambda\|U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega}) \\ &\le c(\|\partial_x^2 (U - P^2_{\mathbf{N},\lambda,\delta,\Omega} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} + \lambda\|U - P^2_{\mathbf{N},\lambda,\delta,\Omega} U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega}) \\ &\le c\Bigg(N_1^{4-2r_1} \|\partial_x^{r_1} U\|^2_{\chi^{(-2+r_1,-2+r_1)},I} \\ &\qquad + (1+\lambda)\left(1 + \frac{1}{\beta^4}\right)(\beta N_2)^{2-r_2} \left\| \partial_x^{r_2} \left((\delta + x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} U \right) \right\|^2_{\omega_{-2+r_2,\beta},\Lambda} \Bigg). \end{aligned} \tag{5.53}$$

We next consider the following mixed inhomogeneous boundary value problem,

$$\begin{cases} \partial_x^4 U(x) + \lambda U(x) = f(x), & x \in \Omega, \\ \partial_x^2 U(-2) = b, & U(-2) = a, \end{cases} \tag{5.54}$$

where the asymptotic behavior of $U(x)$ and $f(x)$, a, b are the same as in (5.44).

Let $\tilde{H}^2_{\hat{\omega}_{0,\gamma,\delta}}(\Omega)$ be the same as before, and

$$V(\Omega) = \{v \mid v \in \tilde{H}^2_{\hat{\omega}_{0,\gamma,\delta}}(\Omega) \text{ and } v(-2) = a\},$$

$$\overline{V}(\Omega) = \{v \mid v \in H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Omega) \text{ and } v(-2) = 0\}.$$

The weak form of (5.54) is to find $U \in V(\Omega)$ such that

$$\mathcal{A}_{\lambda,\delta,\Omega}(U, v) + b(\delta-2)^{-\gamma}\partial_x v(-2) = (f, v)_{\hat{\omega}_{0,\gamma,\delta},\Omega}, \quad \forall\, v \in \overline{V}(\Omega). \tag{5.55}$$

If $\lambda > c_\gamma$, and $\gamma \geq 0$ or $\delta = 2$, then the above problem admits a unique solution.

Next, let

$$V_{\mathbf{N}}(\Omega) = \{\phi \in V(\Omega) \mid \phi|_I \in P_{N_1}(I), \phi|_\Lambda \in \hat{Q}_{N_2}^{(0,\beta,\gamma,\delta)}(\Lambda)\},$$

$$\overline{V}_{\mathbf{N}}(\Omega) = \{\phi \in \overline{V}(\Omega) \mid \phi|_I \in P_{N_1}(I), \phi|_\Lambda \in \hat{Q}_{N_2}^{(0,\beta,\gamma,\delta)}(\Lambda)\}.$$

The multidomain spectral method for solving problem (5.45), is to seek $u_{\mathbf{N}} \in V_{\mathbf{N}}(\Omega)$ such that

$$\mathcal{A}_{\lambda,\delta,\Omega}(u_{\mathbf{N}}, \phi) + b(\delta-2)^{-\gamma}\partial_x \phi(-2) = (f, \phi)_{\hat{\omega}_{0,\gamma,\delta},\Omega}, \quad \forall\, \phi \in \overline{V}_{\mathbf{N}}(\Omega). \tag{5.56}$$

If $\lambda > c_\gamma$, and $\gamma \geq 0$ or $\delta = 2$, then the problem (5.56) has a unique solution.

We now turn to the error estimate of the numerical solution. Let $P^2_{\mathbf{N},\gamma,\delta,\Omega} v(x)$ be the same as in (5.47). We also introduce the auxiliary operator $\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} : V(\Omega) \to V_{\mathbf{N}}(\Omega)$, defined by

$$\mathcal{A}_{\lambda,\delta,\Omega}(\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} v - v, \phi) = 0, \quad \forall\, \phi \in \overline{V}_{\mathbf{N}}(\Omega). \tag{5.57}$$

Indeed, $u_{\mathbf{N}} = \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$. Thus, it remains to estimate the error of $\overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$.

We use Proposition 5.2, with $W(\Omega) = V(\Omega)$, $\overline{W}(\Omega) = \overline{V}(\Omega)$, $W_{\mathbf{N}}(\Omega) = V_{\mathbf{N}}(\Omega)$, $\overline{W}_{\mathbf{N}}(\Omega) = \overline{V}_{\mathbf{N}}(\Omega)$, $v = U$, $w = P^2_{\mathbf{N},\gamma,\delta,\Omega} U$ and ${}_*P^2_{\mathbf{N},\lambda,\delta,\Omega} U = \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U$. Then, by virtue of (5.40), (5.43) and (5.39), we follow the same argument as in the derivation of (5.52) to verify that

$$\begin{aligned} &\frac{1}{2}\|\partial_x^2(U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} + (\lambda - c_\gamma)\|U - \overline{P}^2_{\mathbf{N},\lambda,\delta,\Omega} U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Omega} \\ &\quad \leq \left(\frac{7}{4} + 4\gamma^2 d_{\gamma,\delta}\right)\|\partial_x^2(U - P^2_{\mathbf{N},\gamma,\delta,\Omega} U)\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda} + \left(\lambda + 8\gamma^2 d_{\gamma,\delta} + \gamma^2(\gamma+1)^2\right)\|U - P^2_{\mathbf{N},\gamma,\delta,\Omega} U\|^2_{\hat{\omega}_{0,\gamma,\delta},\Lambda}. \end{aligned} \tag{5.58}$$

Finally, we follow the same line as the derivation of (5.53) to assert if $\lambda > c_\gamma$, $\gamma \geq 0$ and $\delta > 2$ (or $\gamma \leq 0$ and $\delta = 2$), then for integers $r_1, r_2 \geq 2$,

$$\begin{aligned} \|U - u_{\mathbf{N}}\|^2_{H^2_{\hat{\omega}_{0,\gamma,\delta}}(\Omega)} \leq c\Bigg(& N_1^{4-2r_1}\|\partial_x^{r_1} U\|^2_{\chi^{(-2+r_1,-2+r_1)},I} + (1+\lambda) \\ & \times \left(1 + \frac{1}{\beta^4}\right)(\beta N_2)^{2-r_2}\left\|\partial_x^{r_2}\left((\delta+x)^{-\frac{\gamma}{2}} e^{\frac{\beta}{2}x} U\right)\right\|^2_{\omega_{-2+r_2,\beta},\Lambda}\Bigg). \end{aligned} \tag{5.59}$$

Remark 5.3. In the early work of the Legendre–Laguerre multidomain spectral method, one used different weights on I and Λ, respectively; see [16,27]. In this paper, we use the uniform weight $(\delta + x)^{-\gamma}$ on the whole unbounded domain Ω, with which it is much more convenient to deal with high order problems.

Remark 5.4. We may design and analyze the Legendre–Laguerre multidomain spectral method with exact imposition of mixed inhomogeneous boundary conditions, as in Remark 5.2.

6. Numerical results

In this section, we present some numerical results illustrating the efficiency of proposed methods.

Table 1
The errors of algorithm (6.3) with $\beta = 1$.

N	$\gamma = 0$		$\gamma = -2$	
	$\|U - u_N\|_{N,\mathrm{ave}}$	$\|U - u_N\|_{N,\max}$	$\|U - u_N\|_{N,\mathrm{ave}}$	$\|U - u_N\|_{N,\max}$
20	9.93E−3	3.95E−3	2.26E−3	1.95E−3
40	7.19E−4	2.05E−4	7.79E−5	3.24E−5
60	8.26E−5	1.93E−5	1.67E−5	4.63E−6
80	1.28E−5	2.59E−6	6.24E−6	1.50E−6

6.1. *Spectral scheme (5.14)*

We take the basis functions as follows,

$$\begin{aligned} \phi_0(x) &= \delta^{-\frac{\gamma}{2}}\left(1 + \frac{\beta - \gamma\delta^{-1}}{2}x\right)(\delta + x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x}, \\ \phi_1(x) &= \delta^{-\frac{\gamma}{2}} x(\delta + x)^{\frac{\gamma}{2}} e^{-\frac{\beta}{2}x}, \\ \phi_l(x) &= \frac{\beta^2}{l(l-1)} \hat{\mathcal{L}}_l^{(-2,\beta,\gamma,\delta)}(x), \quad 2 \le l \le N. \end{aligned} \tag{6.1}$$

Evidently, $\partial_x^k \phi_l(0) = \delta_{k,l}$ for $0 \le k, l \le 1$, and $\partial_x^k \phi_l(0) = 0$ for $0 \le k \le 1$, $2 \le l \le N$.

We expand the numerical solution as

$$u_N(x) = \sum_{l=2}^{N} \hat{u}_{N,l}\phi_l(x) + b\phi_1(x) + a\phi_0(x) \in V_N(\Lambda).$$

Inserting the above expression into the scheme (5.14) with $\phi = \phi_k$, we obtain

$$\sum_{l=2}^{N} (a_{k,l} - 2\gamma b_{k,l} + \gamma(\gamma+1)d_{k,l} + \lambda g_{k,l})\hat{u}_{N,l} = F_k, \quad 2 \le k \le N, \tag{6.2}$$

where

$$\begin{aligned} &a_{k,l} = (\partial_x^2\phi_l, \partial_x^2\phi_k)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \qquad b_{k,l} = (\partial_x^2\phi_l, \partial_x\phi_k)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \\ &d_{k,l} = (\partial_x^2\phi_l, \phi_k)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \qquad g_{k,l} = (\phi_l, \phi_k)_{\hat{\omega}_{0,\gamma,\delta},\Lambda}, \\ &F_k = (f, \phi_k)_{\hat{\omega}_{0,\gamma,\delta},\Lambda} - b\mathcal{A}_{\lambda,\delta,\Lambda}(\phi_1, \phi_k) - a\mathcal{A}_{\lambda,\delta,\Lambda}(\phi_0, \phi_k). \end{aligned}$$

Next, we set

$$\begin{aligned} &A = (a_{k,l})_{2 \le k,l \le N}, \qquad B = (b_{k,l})_{2 \le k,l \le N}, \qquad D = (d_{k,l})_{2 \le k,l \le N}, \qquad G = (g_{k,l})_{2 \le k,l \le N}, \\ &\mathbf{u} = (\hat{u}_{N,2}, \hat{u}_{N,3}, \ldots, \hat{u}_{N,N})^T, \qquad \mathbf{F} = (F_2, F_3, \ldots, F_N)^T. \end{aligned}$$

Then, we obtain the compact matrix form of (6.2) as follows,

$$(A - 2\gamma B + \gamma(\gamma+1)D + \lambda G)\mathbf{u} = \mathbf{F}. \tag{6.3}$$

Let $x_{La,N,j}$ and $\omega_{La,N,j}$ be the nodes and weights of the Laguerre–Gauss–Radau numerical quadrature on the interval Λ, respectively. We measure the errors of numerical solutions by the following quantities:

$$\begin{aligned} \|U - u_N\|_{N,\mathrm{ave}} &= \left(\sum_{j=1}^{N} (U(x_{La,N,j}) - u_N(x_{La,N,j}))^2 \omega_{La,N,j}\right)^{\frac{1}{2}}, \\ \|U - u_N\|_{N,\max} &= \max_{1 \le j \le N} |U(x_{La,N,j}) - u_N(x_{La,N,j})|. \end{aligned}$$

We now use the algorithm (6.3) to solve problem (5.13) with the test function

$$U(x) = \frac{\sin x}{(x+b)^h}, \quad h \ge 1. \tag{6.4}$$

Clearly, $U \in L^2_{\hat{\omega}_{0,\gamma,\delta}}(\Lambda)$ as long as $\gamma > -2h + 1$.

We first take $\lambda = 9$ and $b = 1$ in actual computation. In this case, $\lambda \ge c_\gamma$ for $\gamma = 0, -2$. Therefore, the error estimate (5.23) is valid. In Table 1, we list the values of $\|U - u_N\|_{N,\mathrm{ave}}$ and $\|U - u_N\|_{N,\max}$, with $h = 3$, $\beta = \delta = 1$ and $\gamma = 0, -2$, vs. the mode N. The numerical results demonstrate the convergence of algorithm (6.3), as predicted by (5.23). They also show

Table 2
The errors of algorithm (6.3) with $\gamma = -2$.

N	$\beta = 1$		$\beta = 2$	
	$\|U-u_N\|_{N,\text{ave}}$	$\|U-u_N\|_{N,\max}$	$\|U-u_N\|_{N,\text{ave}}$	$\|U-u_N\|_{N,\max}$
20	2.26E−3	1.95E−3	5.92E−5	2.32E−5
40	7.79E−5	3.24E−5	2.23E−6	5.81E−7
60	1.67E−5	4.63E−6	4.22E−7	2.04E−7
80	6.24E−6	1.50E−6	1.99E−7	8.82E−8

Table 3
The errors of algorithm (6.3) with $\beta = 2$, $\gamma = -2$.

N	$h = 3$		$h = 4$	
	$\|U-u_N\|_{N,\text{ave}}$	$\|U-u_N\|_{N,\max}$	$\|U-u_N\|_{N,\text{ave}}$	$\|U-u_N\|_{N,\max}$
20	5.92E−5	2.32E−5	1.03E−4	6.33E−5
40	2.23E−6	5.81E−7	1.19E−6	2.96E−7
60	4.22E−7	2.04E−7	3.89E−8	7.65E−9
80	1.99E−7	8.82E−8	4.07E−9	8.42E−10

Table 4
The errors of algorithm (6.3) with $\beta = 2$, $\gamma = -2$.

N	$\lambda = 1$		$\lambda = 9$	
	$\|U-u_N\|_{N,\text{ave}}$	$\|U-u_N\|_{N,\max}$	$\|U-u_N\|_{N,\text{ave}}$	$\|U-u_N\|_{N,\max}$
20	6.97E−5	2.48E−5	5.92E−5	2.32E−5
40	7.74E−6	5.18E−6	2.23E−6	5.81E−7
60	3.14E−6	1.70E−6	4.22E−7	2.04E−7
80	1.60E−6	7.08E−7	1.99E−7	8.82E−8

that the results with $\gamma = -2$ are better than the results with $\gamma = 0$. In fact, the basis functions with $\gamma = -2$ simulate the asymptotic behavior of the test function (6.4) with $h = 3$ more reasonably. In Table 2, we list the values of $\|U - u_N\|_{N,\text{ave}}$ and $\|U - u_N\|_{N,\max}$, with $h = 3$, $\beta = 1, 2$, $\delta = 1$ and $\gamma = -2$, vs. the mode N. They demonstrate the convergence of (6.3) again. They also show that a suitable choice of parameter β could raise the numerical accuracy. In Table 3, we present the values of $\|U - u_N\|_{N,\text{ave}}$ and $\|U - u_N\|_{N,\max}$, with $h = 3, 4$, $\beta = 2$, $\delta = 1$ and $\gamma = -2$, vs. the mode N. They indicate that the smoother the exact solutions, the better the numerical results. This coincides with the theoretical analysis.

Next, we present in Table 4 the values of $\|U - u_N\|_{N,\text{ave}}$ and $\|U - u_N\|_{N,\max}$, with $b = 1$, $h = 3$, $\lambda = 1, 9$, $\beta = 2$, $\delta = 1$ and $\gamma = -2$, vs. the mode N. We find that the results with $\lambda = 9$ are better than the results with $\lambda = 1$. Indeed, $\lambda \geq c_\gamma$ for $\lambda = 9$ and $\gamma = -2$. Thus, the error estimate (5.23) ensures the higher accuracy of (6.3). But the condition $\lambda \geq c_\gamma$ is only a sufficient condition for the convergence. Therefore, algorithm (6.3) with $\lambda = 1$ and $\gamma = -2$ might still convergent, as it is shown in Table 4.

6.2. *Multidomain spectral scheme (5.46)*

We need six kinds of basis functions. The first kind of local basis functions correspond to the subdomains I and Λ respectively. They are of the following forms,

$$\psi_{I,l}(x) = \begin{cases} \dfrac{1}{4(l-2)(l-3)} Y_l^{(2,2)}(x), & x \in I, \\ 0, & \text{otherwise}, \end{cases} \tag{6.5}$$

$$\psi_{\Lambda,l}(x) = \begin{cases} \dfrac{\beta^2}{l(l-1)} \hat{\mathcal{L}}_l^{(-2,\beta,\gamma,\delta)}(x), & x \in \Lambda, \\ 0, & \text{otherwise}. \end{cases} \tag{6.6}$$

Next, we define the basis function for matching the numerical solution at $x = 0$, namely,

$$\sigma(x) = \begin{cases} q^+_{2,2,0,I}(x), & x \in \bar{I}, \\ \phi_0(x), & x \in \bar{\Lambda}. \end{cases} \tag{6.7}$$

Similarly, we define the basis function for matching the first order derivative of the numerical solution at $x = 0$, as

$$\rho(x) = \begin{cases} q^+_{2,2,1,I}(x), & x \in \bar{I}, \\ \phi_1(x), & x \in \bar{\Lambda}. \end{cases} \tag{6.8}$$

Obviously, $\sigma(0) = \partial_x\rho(0) = 1$ and $\partial_x\sigma(0) = \rho(0) = 0$.

Table 5
The errors of algorithm (6.12).

N_1	$\|U-u_{\mathbf{N}}\|_{\mathbf{N},\mathrm{ave}}$	$\|U-u_{\mathbf{N}}\|_{\mathbf{N},\max}$
5	2.84E−2	2.23E−3
10	6.48E−5	3.64E−5
15	4.62E−7	1.85E−7
20	1.91E−7	8.21E−8

Table 6
The maximum errors of GLA and LA approximations.

	$h=-3$	$h=-1$	$h=0$	$h=1$	$h=3$
GLA	9.90E−12	3.19E−14	1.89E−14	1.43E−14	7.99E−15
LA	1.12E−07	7.11E−10	2.31E−10	1.66E−10	3.20E−11

Furthermore, we take the basis function for matching the numerical solution at $x=-2$, namely,

$$\xi(x)=\begin{cases} q^{-}_{2,2,0,I}(x), & x\in\bar{I},\\ 0, & \text{otherwise}.\end{cases} \tag{6.9}$$

We also take the basis function for matching the first order derivative of the numerical solution at $x=-2$, as

$$\zeta(x)=\begin{cases} q^{-}_{2,2,1,I}(x), & x\in\bar{I},\\ 0, & \text{otherwise}.\end{cases} \tag{6.10}$$

Clearly, $\xi(-2)=\partial_x\zeta(-2)=1$ and $\partial_x\xi(-2)=\zeta(-2)=0$.

In actual computation, we expand the numerical solution as

$$u_{\mathbf{N}}(x)=\sum_{l=4}^{N_1}\hat{u}_{I,l}\psi_{I,l}(x)+\sum_{l=2}^{N_2}\hat{u}_{\Lambda,l}\psi_{\Lambda,l}(x)+u_{\mathbf{N}}(0)\sigma(x)+\partial_x u_{\mathbf{N}}(0)\rho(x)+a\xi(x)+b\zeta(x)\in V_N(\Omega). \tag{6.11}$$

Inserting (6.11) into (5.46), we obtain

$$\mathcal{A}_{\lambda,\delta,\Omega}(u_{\mathbf{N}},\phi)=(f,\phi)_{\hat{\omega}_{0,\gamma,\delta},\Omega},\quad \forall\phi\in\overline{V}_N(\Omega) \tag{6.12}$$

where the test functions ϕ stand for the basis functions given by (6.5)–(6.10).

Let $x_{Le,N_1,j}$ and $\omega_{Le,N_1,j}$ be the nodes and weights of the Legendre–Gauss–Lobatto numerical quadrature on the interval $(-1,1)$, respectively. We measure the numerical errors on the whole domain Ω, by the following quantities:

$$\|U-u_{\mathbf{N}}\|_{\mathbf{N},\mathrm{ave}}=\left(\sum_{j=0}^{N_1}(U(x_{Le,N_1,j}-1)-u_{\mathbf{N}}(x_{Le,N_1,j}-1))^2\omega_{Le,N_1,j}+\sum_{j=1}^{N_2}(U(x_{La,N_2,j})-u_{\mathbf{N}}(x_{La,N_2,j}))^2\omega_{La,N_2,j}\right)^{\frac{1}{2}},$$

$$\|U-u_{\mathbf{N}}\|_{\mathbf{N},\max}=\max\Big(\max_{0\le j\le N_1}|U(x_{Le,N_1,j}-1)-u_{\mathbf{N}}(x_{Le,N_1,j}-1)|,\ \max_{1\le j\le N_2}|U(x_{La,N_2,j})-u_{\mathbf{N}}(x_{La,N_2,j})|\Big).$$

We now use (6.12) with $\beta=2$, $\delta=3$ and $\gamma=-2$ to solve problem (5.44) with $\lambda=9$ and the test function (6.4) with $b=h=3$. In Table 5, we list the values of $\|U-u_{\mathbf{N}}\|_{\mathbf{N},\mathrm{ave}}$ and $\|U-u_{\mathbf{N}}\|_{\mathbf{N},\max}$, with $N_2=4N_1$, vs. the modes N_1. They illustrate the high accuracy of (6.12).

Remark 6.1. In actual computation, we may adopt the usual orthogonal approximation using the Laguerre polynomials. This process also possesses the spectral accuracy in the global weighted norm. But because of the existence of the weight function e^{-x}, the small numerical errors in the global weighted norm do not imply the small numerical errors in the point-wise norm. Conversely, in the new generalized Laguerre orthogonal approximation of this work, we could adjust the parameter α for fitting the value of approximated functions at $x=0$ exactly, and the parameter γ for simulating the asymptotic behaviors at infinity reasonably. Therefore, it provides much better numerical results.

We now consider the function $\frac{\sin kx}{(1+x)^h}$, which belongs to the space $L^2_{\hat{\omega}_{\alpha,\gamma,1}}(\Lambda)$ with $\gamma>-2h+\alpha+1$, and oscillates as x increases. Let $\eta_{N,j}$ be the nodes of the corresponding interpolations, $0\le j\le N$. We measure the numerical errors by the maximum norm $\|v\|_{N,\max}=\max_{0\le j\le N}|v(\eta_{N,j})|$. In actual computation, we take $m=1$ and the mode $N=10$.

In Table 6, we list the errors in the maximum norm of the usual Laguerre orthogonal approximation (labeled by LA) and our new generalized orthogonal approximation (labeled by GLA), in which we take $\alpha=-1$, $\beta=1$ and $\gamma=-2h+\frac{1}{2}$. We find that the approximation errors in the maximum norm of our new approach are much smaller than the usual Laguerre orthogonal approximation, no matter the test function decays ($h>0$) or grows up ($h<0$) at infinity.

On the other hand, we could also use the standard Chebyshev rational orthogonal approximation for the functions decaying to zero at infinity; cf. [28]. It has the high accuracy as the generalized orthogonal approximation. But it is better to use the latter for the functions growing up at infinity. For instance, for the above test function with $h=-3$ and $h=-1$, the maximum errors of the standard Chebyshev rational orthogonal approximation are 8.38E−9 and 1.95E−13, respectively.

7. Concluding remarks

In this paper, we introduced the new Laguerre orthogonal approximation with the weight function $x^{\alpha}(\delta+x)^{-\gamma}$, $\delta>0$, α and γ being any real numbers. We also proposed the corresponding quasi-orthogonal approximation and the related Laguerre–Gauss–Radau type interpolation. By adjusting the parameters α and γ suitably, they not only fit the boundary conditions of underlying problems at the fixed boundary exactly, but also match the asymptotic behaviors at infinity reasonably. Moreover, the parameter γ leads to the flexibility for the multidomain spectral method. On the other hand, we used the base functions with the scaling parameter β in the orthogonal approximation, i.e., $(\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\mathscr{L}_l^{(\alpha,\beta)}(x)$. A suitable choice of β could improve the accuracy. These results are very applicable to spectral, pseudospectral and collocation methods for high order problems with mixed inhomogeneous boundary conditions, as well as the related multidomain methods.

As examples of applications, we provided the spectral and multidomain spectral methods for four model problems, and proved their spectral accuracy. The numerical results demonstrated their high efficiency and coincided with the analysis very well. They also showed that the proposed algorithm with suitable parameters α and γ could deal with different underlying problems properly, while the suitable parameter β improves the numerical accuracy. The techniques developed in this work are also useful for other problems, such as problems defined on certain multiple-dimensional unbounded domains, as well as some exterior problems.

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Appendix

Proof of (3.12). We have from (3.11) that

$$\begin{aligned}\partial_x^k v_{b,m,\gamma,\delta,\Lambda}(x) &= \sum_{l=0}^{k} C_k^l \partial_x^l\left((\delta+x)^{\frac{\gamma}{2}}e^{-\frac{\beta}{2}x}\right)\left(\sum_{j=k-l}^{m-1}\frac{j!}{(j-k+l)!}a_j(v)x^{j-k+l}\right)\\ &= \sum_{l=0}^{k} C_k^l\left(\sum_{\mu=0}^{l}C_l^{\mu}\partial_x^{\mu}\left((\delta+x)^{\frac{\gamma}{2}}\right)\partial_x^{l-\mu}\left(e^{-\frac{\beta}{2}x}\right)\right)\left(\sum_{j=k-l}^{m-1}\frac{j!}{(j-k+l)!}a_j(v)x^{j-k+l}\right).\end{aligned}$$

Let $\lambda_0(\gamma)=1$ and $\lambda_\mu(\gamma)=\prod_{\nu=0}^{\mu-1}(\frac{\gamma}{2}-\nu)$ for $\mu\geq 1$. Then, a direct calculation shows that

$$\begin{aligned}\partial_x^k v_{b,m,\gamma,\delta,\Lambda}(0) &= \sum_{l=0}^{k} C_k^l\left(\sum_{\mu=0}^{l}C_l^{\mu}\delta^{\frac{\gamma}{2}-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{l-\mu}\right)(k-l)!a_{k-l}(v)\\ &= \sum_{l=0}^{k} C_k^l\left(\sum_{\mu=0}^{l}C_l^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{l-\mu}\right)\left(\sum_{i=0}^{k-l}(-1)^iC_{k-l}^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i\right.\right.\\ &\quad\left.\left.+\frac{3}{2}\theta_i\gamma\delta^{-2}\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)+\frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{k-l-i}v(0)\right).\end{aligned}\tag{A.1}$$

Clearly, the equality (A.1) with $k=0$ gives $v_{b,m,\gamma,\delta,\Lambda}(0)=v(0)$ directly.

Next, we put $k=1$ in (A.1). Thanks to $\theta_0=\theta_1=\sigma_0=\sigma_1=0$, a direct calculation leads to that

$$\begin{aligned}\partial_x v_{b,m,\gamma,\delta,\Lambda}(0) &= \sum_{i=0}^{1}(-1)^iC_1^i\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i\partial_x^{1-i}v(0)+\sum_{\mu=0}^{1}C_1^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{1-\mu}v(0)\\ &= \partial_x v(0)-\frac{\gamma\delta^{-1}-\beta}{2}v(0)-\frac{\beta}{2}v(0)+\frac{\gamma}{2}\delta^{-1}v(0)=\partial_x v(0).\end{aligned}$$

Further, we put $k=2$ in (A.1). Due to $\theta_0=\theta_1=\theta_2=\sigma_0=\sigma_1=0$, we deduce that

$$\partial_x^2 v_{b,m,\gamma,\delta,\Lambda}(0) = \sum_{i=0}^{2}(-1)^iC_2^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i+\frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{2-i}v(0)$$

$$+2\sum_{\mu=0}^{1} C_1^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{1-\mu}\left(\sum_{i=0}^{1}(-1)^i C_1^i\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i\partial_x^{1-i}v(0)\right)$$
$$+\sum_{\mu=0}^{2} C_2^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{2-\mu} v(0). \tag{A.2}$$

Since $\sigma_2 = 1$, we obtain

$$\sum_{i=0}^{2}(-1)^i C_2^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i+\frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{2-i}v(0)$$
$$=\partial_x^2 v(0)-(\gamma\delta^{-1}-\beta)\partial_x v(0)+\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^2+\frac{1}{2}\gamma\delta^{-2}\right)v(0). \tag{A.3}$$

Moreover, we verify that

$$\sum_{\mu=0}^{1} C_1^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{1-\mu}=\frac{1}{2}(\gamma\delta^{-1}-\beta), \tag{A.4}$$

$$\sum_{i=0}^{1}(-1)^i C_1^i\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i\partial_x^{1-i}v(0)=\partial_x v(0)-\frac{\gamma\delta^{-1}-\beta}{2}v(0), \tag{A.5}$$

$$\begin{aligned}\sum_{\mu=0}^{2} C_2^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{2-\mu} &= \frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)\delta^{-2}-\frac{\gamma\beta}{2}\delta^{-1}+\left(\frac{\beta}{2}\right)^2\\ &= \left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^2-\frac{1}{2}\gamma\delta^{-2}.\end{aligned} \tag{A.6}$$

By inserting (A.3)–(A.6) into (A.2), we find $\partial_x^2 v_{b,m,\gamma,\delta,\Lambda}(0)=\partial_x^2 v(0)$.

Finally, we have from the equality (A.1) with $k = 3$ that

$$\begin{aligned}\partial_x^3 v_{b,m,\gamma,\delta,\Lambda}(0) = {} & \sum_{i=0}^{3}(-1)^i C_3^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i+\frac{3}{4}\theta_i\gamma\delta^{-2}(\gamma\delta^{-1}-\beta)+\frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{3-i}v(0)\\ &+3\sum_{\mu=0}^{1} C_1^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{1-\mu}\left(\sum_{i=0}^{2}(-1)^i C_2^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i+\frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{2-i}v(0)\right)\\ &+3\sum_{\mu=0}^{2} C_2^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{2-\mu}\left(\sum_{i=0}^{1}(-1)^i C_1^i\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i\partial_x^{1-i}v(0)\right)\\ &+\sum_{\mu=0}^{3} C_3^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{3-\mu} v(0).\end{aligned} \tag{A.7}$$

It can be shown that

$$\begin{aligned}&\sum_{i=0}^{3}(-1)^i C_3^i\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^i+\frac{3}{4}\theta_i\gamma\delta^{-2}(\gamma\delta^{-1}-\beta)+\frac{1}{2}(i-1)\sigma_i\gamma\delta^{-i}\right)\partial_x^{3-i}v(0)\\ &=\partial_x^3 v(0)-3\frac{\gamma\delta^{-1}-\beta}{2}\partial_x^2 v(0)+3\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^2+\frac{\gamma}{2}\delta^{-2}\right)\partial_x v(0)-\left(\left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^3\right.\\ &\quad\left.+\frac{3}{4}\gamma\delta^{-2}(\gamma\delta^{-1}-\beta)+\gamma\delta^{-3}\right)v(0),\end{aligned} \tag{A.8}$$

$$\begin{aligned}\sum_{\mu=0}^{3} C_3^{\mu}\delta^{-\mu}\lambda_\mu(\gamma)\left(-\frac{\beta}{2}\right)^{3-\mu} &= \frac{\gamma}{2}\left(\frac{\gamma}{2}-1\right)\left(\frac{\gamma}{2}-2\right)\delta^{-3}-\frac{3}{4}\gamma\left(\frac{\gamma}{2}-1\right)\beta\delta^{-2}+\frac{3}{2}\gamma\left(\frac{\beta}{2}\right)^2\delta^{-1}+\left(-\frac{\beta}{2}\right)^3\\ &= \left(\frac{\gamma\delta^{-1}-\beta}{2}\right)^3-\gamma\left(\frac{3}{4}\gamma-1\right)\delta^{-3}+\frac{3}{4}\gamma\beta\delta^{-2}.\end{aligned} \tag{A.9}$$

By substituting (A.3)–(A.6), (A.8) and (A.9) into (A.7), we assert that $\partial_x^3 v_{b,m,\gamma,\delta,\Lambda}(0) = \partial_x^3 v(0)$. This completes the proof of (3.12). $\square$

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