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A New Collocation Scheme Using Non-polynomial Basis Functions

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Abstract In this paper, we construct a set of non-polynomial basis functions from a generalised Birkhoff interpolation problem involving the operator: $\mathscr{L}_\lambda = d^2/dx^2 - \lambda^2$ with constant λ. With a direct inverting the operator, the basis can be pre-computed in a fast and stable manner. This leads to new collocation schemes for general second-order boundary value problems with (i) the matrix corresponding to the operator $\mathscr{L}_\lambda$ being identity; (ii) well-conditioned linear systems and (iii) exact imposition of various boundary conditions. This also provides efficient solvers for time-dependent nonlinear problems. Moreover, we can show that the new basis has the approximability to general functions in Sobolev spaces as good as orthogonal polynomials.

Keywords Generalised Birkhoff interpolation problem · Non-polynomial basis · Well-conditioned collocation methods

Mathematics Subject Classification 65N35 · 65E05 · 65M70 · 41A05 · 41A10 · 41A25

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1 Introduction

The spectral method employs global orthogonal systems, e.g., Fourier complex exponentials and orthogonal polynomials, as basis functions. It enjoys the remarkable "spectral accuracy" (i.e., exponential order of convergence), if the underlying solution is smooth (and sufficiently periodic in the Fourier case). Perhaps, the Fourier functions $\{e^{\mathrm{i}kx}\}$ are most desirable due to (i) the availability of Fast Fourier transform (FFT), and (ii) the resultant of a diagonal linear system for any linear differential operator with constant coefficients. However, it is well known that the Fourier method is only applicable to periodic problems, as it has a very poor convergence rate in non-periodic cases, owing to the Gibbs phenomena. In such cases, one should use polynomial-based spectral algorithms using e.g., Chebyshev, Legendre or Jacobi polynomials as basis functions, which are eigenfunctions of singular Sturm–Liouville problems.

It is noteworthy that spectral-Galerkin methods based on compact combinations of orthogonal polynomials [26], defined as generalised Jacobi polynomials [16,17], leading to optimal algorithms in terms of both conditioning and finite banded coefficient matrices for linear operators with constant coefficients. For example, the integrated Legendre polynomial basis leads to a diagonal stiffness matrix and peta-diagonal mass matrix for the differential operator: $\mathscr{L}_\lambda[u] = u'' - \lambda^2 u$ (with constant λ and homogeneous Dirichlet boundary conditions). In practice, it is advantageous to have a Fourier-like polynomial basis. In [28], a polynomial basis mutually orthogonal in both L^2- and H^1-inner products was constructed from solving discrete eigen-value problems. Such an eigen-basis led to efficient spectral-element approaches on structured meshes in [38]. We remark that Livermore [22] considered Galerkin-orthogonal polynomials for spectral approximation in a more general setting.

The spectral collocation method is implemented in physical space, and has notable advantages over the spectral-Galerkin method using modal basis in dealing with variable coefficient and nonlinear problems. However, the practitioners are plagued with the ill-conditioning of the resulted linear systems. The construction of suitable preconditioners is an important means to circumvent this barrier. Significant attempts include preconditioning by low-order finite difference or finite elements (see e.g., [2,3,9,10,20,21]), or by integration (see, e.g., [4,6,7,12,18,32,33]). In particular, from a suitable Birkhoff interpolation problem (cf. [5,23,37]), a new polynomial basis could be constructed which led to well-conditioned collocation schemes for boundary value problems. We also point out that a useful approach pertinent to integration preconditioning is the spectral integration method (cf. [11,15,36]) via recasting a differential form into an integral form, and then approximating the solution by orthogonal polynomials. Various interesting variants can be found in e.g., [13,14,24,25].

The main purpose of this paper is to pursuit new Fourier-like basis (mimic to the Galerkin setting in [22,28]) for the collocation scheme. The significance is that under such a basis, the collocation matrix of $\mathscr{L}_\lambda[u]$ for various boundary conditions becomes identity matrix (see (2.22) below). In distinctive contrast to [22,28], the new basis herein is non-polynomial and builds in the parameter λ intrinsically (e.g., the basis is highly oscillatory if $\lambda = \mathrm{i}k$ for $k \gg 1$). We outline below some other important features of this development.

(i) Using the notion of Birkhoff interpolation as with [32], we construct a new basis $\{Q_j\}$ from a (generalized) Birkhoff interpolation problem (see Theorem 2.2), which can be explicitly obtained by inverting the operator $\mathscr{L}_\lambda$. This allows for rapid evaluation of the new basis by using a recursive convolution algorithm (see, e.g., [1]) in the offline stage (cf. Subsect. 3.3). We can also show that the new basis has the approximability to general functions in Sobolev spaces as good as orthogonal polynomials.

(ii) It is known that for time-dependent nonlinear problems, semi-implicit time marching schemes (i.e., treating nonlinear terms explicitly, but linear terms implicitly) are preferable in practice. This often leads to solving elliptic boundary value problems with parameter depending on $1/\tau$ (τ is the time-stepping size) at each time step. For problems in one spatial dimension, our collocation scheme provides an explicit solver so that the semi-implicit scheme appears like an "explicit" unconditionally stable scheme (cf. Subsect. 5.2).

(iii) Equipped with the new basis, well-conditioned collocation methods can be constructed for general second-order boundary value problems (BVPs) with exact imposition of various boundary conditions. Different from [12,18,32] (where the preconditioners are implemented on the highest derivative to make the corresponding matrix diagonal, so the condition number for e.g., $\mathscr{L}_\lambda$ still depends on $|\lambda|$, e.g., the wavenumber), we can choose the parameter λ and the conditioning of the system becomes independent of the large or small parameter in the BVP. It is noteworthy that the limiting case $\lambda \to 0^+$, corresponds to the approach in [32], which therefore aims at diagonalising the matrix of the second-order derivative, and the new basis functions are polynomials. We also reiterate that the technique and procedure for $\lambda \neq 0$ herein is very different and much more involved.

The rest of the paper is organised as follows. In Sect. 2, we introduce the new basis from the perspective of generalised Birkhoff interpolation, and present its important properties. We then describe a rapid and stable way to compute the new basis functions in Sect. 3. We show in Sect. 4 the spectral approximation properties of the new basis. In the last section, we apply the new collocation methods to solutions of various problems including the Helmholtz equation with highly oscillatory solutions, Burgers equation, and two-dimensional elliptic problems. We demonstrate the high accuracy and efficiency of the proposed approach.

2 New Non-polynomial Basis Functions

2.1 A General Setup

Consider the second-order linear differential operator:

$$\mathscr{L}_\lambda[u](x) := u''(x) - \lambda^2 u(x), \quad x \in \Lambda := (-1, 1), \tag{2.1}$$

where the constant $\lambda \neq 0$. Let $\mathscr{B}_\pm$ be two linear boundary operators imposed on the boundary-value problem (BVP):

$$\begin{cases} \mathscr{L}_\lambda[u](x) = f(x), & x \in \Lambda; \\ \mathscr{B}_-[u] = g_-, \quad \mathscr{B}_+[u] = g_+, \end{cases} \tag{2.2}$$

such that it has a unique solution for given integrable f and constants $g_\pm$. Typically, we consider

$$\mathscr{B}_\pm[u] = u(\pm 1) \ \text{ or } \ \mathscr{B}_\pm[u] = u'(\pm 1) \ \text{ or } \ \mathscr{B}_-[u] = u(-1), \quad \mathscr{B}_+[u] = u'(1) + \eta\, u(1), \tag{2.3}$$

where η is a nonzero constant. For notational convenience, we define the integral operators:

$$\mathscr{I}_\lambda^-[f](x) = \frac{1}{2\lambda}\int_{-1}^{x} e^{-\lambda(x-y)} f(y)\,\mathrm{d}y; \quad \mathscr{I}_\lambda^+[f](x) = \frac{1}{2\lambda}\int_{x}^{1} e^{-\lambda(y-x)} f(y)\,\mathrm{d}y. \tag{2.4}$$

It is evident that

$$\left(\mathcal{I}_\lambda^-[f]\right)' = -\lambda\,\mathcal{I}_\lambda^-[f] + \frac{1}{2\lambda}f; \quad \left(\mathcal{I}_\lambda^+[f]\right)' = \lambda\,\mathcal{I}_\lambda^+[f] - \frac{1}{2\lambda}f. \tag{2.5}$$

The following solution formula of (2.2) can be derived from fundamental solution techniques of ordinary differential equations in any standard textbook.

Proposition 2.1 *The solution of the BVP* (2.2) *with* $\lambda \neq 0$ *is given by*

$$u(x) = \mathcal{L}_\lambda^{-1}[f](x) = C_1\, e^{\lambda(1+x)} + C_2\, e^{-\lambda(1+x)} - \mathcal{I}_\lambda^-[f](x) - \mathcal{I}_\lambda^+[f](x), \tag{2.6}$$

where the constants C_1 *and* C_2 *are uniquely determined by the boundary conditions in* (2.2). *In particular, we have the formulas of* C_1 *and* C_2 *for the typical boundary conditions in* (2.3) *given in Appendix 6.*

Let $\{x_j, \omega_j\}_{j=0}^N$ (with $x_0 = -1$ and $x_N = 1$) be a set of Jacobi–Gauss–Lobatto (JGL) points and quadrature weights (see Appendix 7). Assume that $\{x_j\}$ are arranged in ascending order. Denote by $\mathcal{P}_N$ be the set of all polynomials of degree at most N. Let $\{l_j\}$ be the Lagrange interpolating basis polynomials associated with interior JGL points $\{x_j\}_{j=1}^{N-1}$ (i.e., zeros of $\partial_x P_N^{(\alpha,\beta)}(x)$ with $\alpha, \beta > -1$):

$$l_j \in \mathcal{P}_{N-2}, \quad l_j(x_i) = \delta_{ij}, \quad 1 \le i, j \le N-1. \tag{2.7}$$

We look for a new basis $\{Q_j\}_{j=0}^N$ such that

- for $1 \le j \le N-1$,

$$\mathscr{L}_\lambda[Q_j](x) = l_j(x), \quad x \in \Lambda; \quad \mathscr{B}_\pm[Q_j] = 0; \tag{2.8}$$

- for $j = 0, N$,

$$\mathscr{L}_\lambda[Q_0](x) = 0, \quad x \in \Lambda; \quad \mathscr{B}_-[Q_0] = 1, \quad \mathscr{B}_+[Q_0] = 0, \tag{2.9}$$

and

$$\mathscr{L}_\lambda[Q_N](x) = 0, \quad x \in \Lambda; \quad \mathscr{B}_-[Q_N] = 0, \quad \mathscr{B}_+[Q_N] = 1. \tag{2.10}$$

The so-defined $\{Q_j\}_{j=0}^N$ can be uniquely determined by the solution formulas in Proposition 2.1 straightforwardly, as summarized below.

Theorem 2.1 *Let* $\{Q_j\}_{j=0}^N$ *be the basis defined in* (2.8)–(2.10). *Then we have*

- *for* $1 \le j \le N-1$,

$$Q_j(x) = C_{1j}\, e^{\lambda(1+x)} + C_{2j}\, e^{-\lambda(1+x)} - \mathcal{I}_\lambda^-[l_j](x) - \mathcal{I}_\lambda^+[l_j](x), \tag{2.11}$$

where the constants C_{1j} *and* C_{2j} *can be computed by the formulas of* C_1 *and* C_2 *in Proposition 2.1 and Appendix 6 with* $g_\pm = 0$ *and* $f = l_j$.

- *for* $j = 0$,

$$Q_0(x) = C_{10}\, e^{\lambda(1+x)} + C_{20}\, e^{-\lambda(1+x)}, \tag{2.12}$$

where C_{10} *and* C_{20} *can be computed by the formulas of* C_1 *and* C_2 *in Proposition 2.1 and Appendix 6 with* $g_- = 1$ *and* $g_+ = f = 0$.

- *for* $j = N$,

$$Q_N(x) = C_{1N}\, e^{\lambda(1+x)} + C_{2N}\, e^{-\lambda(1+x)}, \tag{2.13}$$

where C_{1N} *and* C_{2N} *can be computed by the formulas of* C_1 *and* C_2 *in Proposition 2.1 and Appendix 6 with* $g_+ = 1$ *and* $g_- = f = 0$.

Correspondingly, we define the finite dimensional vector space

$$\mathcal{Q}_N := \text{span}\{Q_0, Q_1, \ldots, Q_N\}. \tag{2.14}$$

Remarkably, the new basis has direct bearing on a generalized Birkhoff interpolation problem which we highlight below.

Theorem 2.2 *Let $\{x_j\}$ be the JGL points. Consider the interpolation problem: given $u \in C^2(\bar{\Lambda})$, find $q \in \mathcal{Q}_N$ such that*

$$\mathscr{L}_\lambda[q](x_j) = \mathscr{L}_\lambda[u](x_j), \quad 1 \le j \le N-1; \quad \mathscr{B}_\pm[q] = \mathscr{B}_\pm[u]. \tag{2.15}$$

Then the interpolant of u has the representation:

$$(\mathbb{I}_N^B u)(x) = q(x) = \mathscr{B}_-[u]\, Q_0(x) + \sum_{j=1}^{N-1} \mathscr{L}_\lambda[u](x_j) Q_j(x) + \mathscr{B}_+[u]\, Q_N(x), \tag{2.16}$$

where $\mathbb{I}_N^B : C^2(\bar{\Lambda}) \to \mathcal{Q}_N$ is the corresponding interpolation operator.

Proof Taking $x = x_i$ for $1 \le i \le N-1$ in (2.8)–(2.10), leads to

$$\begin{cases} \mathscr{L}_\lambda[Q_j](x_i) = \delta_{ij}, \quad 1 \le i, j \le N-1, \quad \mathscr{B}_\pm[Q_j] = 0, \quad 1 \le j \le N-1; \\ \mathscr{B}_-[Q_0] = 1, \quad \mathscr{B}_+[Q_0] = 0, \quad \mathscr{L}_\lambda[Q_0](x_i) = 0, \quad 1 \le i \le N-1; \\ \mathscr{B}_-[Q_N] = 0, \quad \mathscr{B}_+[Q_N] = 1, \quad \mathscr{L}_\lambda[Q_N](x_i) = 0, \quad 1 \le i \le N-1. \end{cases} \tag{2.17}$$

One verifies readily from the interpolating properties in (2.17) that q given by (2.16) satisfies (2.15). It remains to show the uniqueness of q. Assume that $p \in \mathcal{Q}_N$ interpolates the same set of data. Denoting $e := q - p$, we have

$$\mathscr{L}_\lambda[e](x_j) = 0, \quad 1 \le j \le N-1; \quad \mathscr{B}_\pm[e] = 0. \tag{2.18}$$

Since $e \in \mathcal{Q}_N$, we infer from (2.17) and (2.18) that $e \equiv 0$.

Some remarks are in order.

(i) The interpolant q interpolates $\mathscr{L}_\lambda[u]$ at the interior JGL points, rather than the function or derivative values in the usual Birkhoff interpolation (cf. [23]). Moreover, the interpolant q is not a polynomial, if $\lambda \neq 0$. In view of these, we dub (2.15) as a generalized Birkhoff interpolation.

(ii) The interpolant q perfectly matches u at the boundary $x = \pm 1$, which allows for the exact imposition of boundary conditions when u is approximated by q.

2.2 Important Properties

Let $\{h_j\}$ be the Lagrange interpolating basis polynomials associated with the JGL points $\{x_j\}_{j=0}^N$, that is,

$$h_j \in \mathcal{P}_N, \quad h_j(x_i) = \delta_{ij}, \quad 0 \le i, j \le N, \tag{2.19}$$

which should be in contrast with $\{l_j\}$ defined in (2.7) (note: $\{l_j\}$ only involve the interior JGL points). Let $\mathbb{I}_N$ be the corresponding Lagrange interpolation operator (see Appendix 7). Introduce the k-th order differentiation and integration matrices

$$\boldsymbol{D}^{(k)} = \big(d_{ij}^{(k)}\big)_{0\le i,j\le N}, \quad \boldsymbol{D}_{\text{in}}^{(k)} = \big(d_{ij}^{(k)}\big)_{1\le i,j\le N-1} \quad \text{where } d_{ij}^{(k)} = h_j^{(k)}(x_i), \tag{2.20}$$

$$\boldsymbol{Q}^{(k)} = \big(q_{ij}^{(k)}\big)_{0\le i,j\le N}, \quad \boldsymbol{Q}_{\text{in}}^{(k)} = \big(q_{ij}^{(k)}\big)_{1\le i,j\le N-1} \quad \text{where } q_{ij}^{(k)} = Q_j^{(k)}(x_i), \tag{2.21}$$

where $\boldsymbol{D} = \boldsymbol{D}^{(1)}$, $\boldsymbol{D}^{(0)} = \boldsymbol{I}_{N+1}$, $\boldsymbol{Q} = \boldsymbol{Q}^{(0)}$, and $\boldsymbol{Q}_{\text{in}} = \boldsymbol{Q}^{(0)}_{\text{in}}$.

Theorem 2.3 *Let $\boldsymbol{I}_{N-1}$ be the identity matrix of order $N-1$. We have*

$$\boldsymbol{Q}^{(2)}_{\text{in}} - \lambda^2 \boldsymbol{Q}_{\text{in}} = \boldsymbol{I}_{N-1}. \tag{2.22}$$

Let $\mathbb{I}_N$ be the JGL polynomial interpolation operator on $(N+1)$-JGL points $\{x_j\}_{j=0}^N$ (cf. (7.11)). Define the matrices $\boldsymbol{L} = (c_{ij})_{0\le i,j\le N}$ and $\boldsymbol{\mathcal{E}} = (\varepsilon_{ij})_{0\le i,j\le N}$, where for $0 \le j \le N$,

$$c_{ij} := \begin{cases} \mathscr{B}_-[h_j], & i = 0, \\ \mathscr{L}_\lambda[h_j](x_i), & 1 \le i \le N-1, \\ \mathscr{B}_+[h_j], & i = N; \end{cases} \quad \varepsilon_{ij} := \begin{cases} \mathscr{B}_-[\mathbb{I}_N Q_j - Q_j], & i = 0, \\ \mathscr{L}_\lambda[\mathbb{I}_N Q_j - Q_j](x_i), & 1 \le i \le N-1, \\ \mathscr{B}_+[\mathbb{I}_N Q_j - Q_j], & i = N. \end{cases}$$

Then we have

$$\boldsymbol{Q}\boldsymbol{L} = \boldsymbol{I}_{N+1} + \boldsymbol{\mathcal{E}} \approx \boldsymbol{I}_{N+1}. \tag{2.23}$$

Proof The identity (2.22) follows from (2.17) straightforwardly.

We now turn to (2.23). By the definition of $\mathbb{I}_N$, we can write

$$\sum_{l=0}^{N} Q_j(x_l) h_l(x) = \mathbb{I}_N Q_j(x) = \big[\mathbb{I}_N Q_j - Q_j\big](x) + Q_j(x), \quad 0 \le j \le N. \tag{2.24}$$

Acting the linear operators $\mathscr{L}_\lambda$ and $\mathscr{B}_\pm$ on (2.24) leads to

$$\sum_{l=0}^{N} Q_j(x_l) \mathscr{B}_-[h_l] = \mathscr{B}_-\big[\mathbb{I}_N Q_j - Q_j\big] + \mathscr{B}_-[Q_j], \tag{2.25}$$

$$\sum_{l=0}^{N} Q_j(x_l) \mathscr{L}_\lambda[h_l](x) = \mathscr{L}_\lambda\big[\mathbb{I}_N Q_j - Q_j\big](x) + \mathscr{L}_\lambda[Q_j](x), \tag{2.26}$$

$$\sum_{l=0}^{N} Q_j(x_l) \mathscr{B}_+[h_l] = \mathscr{B}_+\big[\mathbb{I}_N Q_j - Q_j\big] + \mathscr{B}_+[Q_j]. \tag{2.27}$$

Thus, taking $x = x_i$ in (2.26), and using (2.17), we obtain the matrix identity in (2.23).

We next show that the entries $\varepsilon_{ij} \approx 0$ for $N \gg 1$. Note that $\{\varepsilon_{ij}\}$ involve up to second derivatives of the JGL interpolation errors of $\{Q_j\}$ at the JGL points $\{x_i\}$. Noting from Theorem 2.1 that for fixed $\lambda \ne 0$, $\{Q_j\}$ are analytic functions, we can characterize the analyticity by the Bernstein ellipse $\mathscr{E}_\rho$ (where $\rho > 1$ is the sum of two semi-axises, see e.g., [8]). Then by [34, Theorem. 4.1], we have the exponential convergence:

$$|\varepsilon_{ij}| \le \frac{CM}{\rho^N}, \quad 0 \le i, j \le N, \tag{2.28}$$

where C is a constant multiple of certain algebraic power of N, and $M = \max_j \max_{z\in\mathscr{E}_\rho} |Q_j(z)|$. Therefore, $\boldsymbol{Q}\boldsymbol{L}$ is nearly an identity matrix for large N.

Observe that $\boldsymbol{L}$ is the coefficient matrix of the system resulted from the usual collocation method for (2.2). More precisely, given $f \in C(\Lambda)$, the collocation scheme for (2.2) is to find $u_N \in \mathcal{P}_N$ such that

$$\begin{cases} \mathscr{L}_\lambda[u_N](x_j) = f(x_j), \quad 1 \le j \le N-1; \\ \mathscr{B}_-[u_N] = g_-, \quad \mathscr{B}_+[u_N] = g_+. \end{cases} \tag{2.29}$$

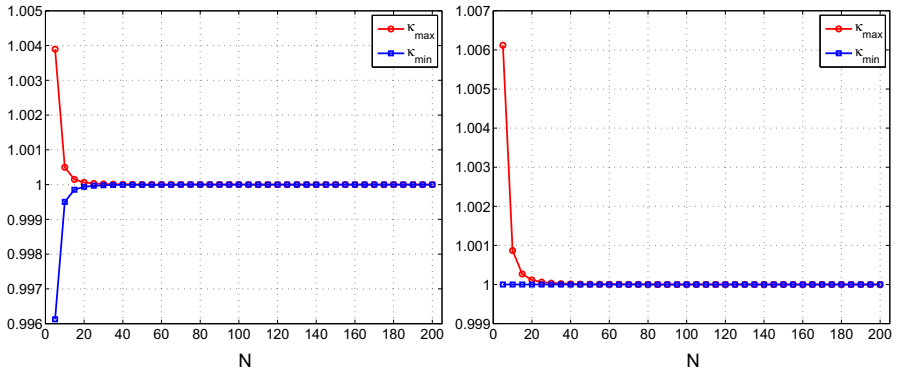

Fig. 1 Distribution of the largest and smallest eigenvalues of $\boldsymbol{QL}$ for various N with $\lambda = 1$. *Left* $\alpha = \beta = -1/2$; *Right* $\alpha = \beta = 0$

Under the Lagrange polynomial basis $\{h_j\}$, the matrix form is

$$\boldsymbol{L}\,\boldsymbol{u} = \boldsymbol{f}, \tag{2.30}$$

where

$$\boldsymbol{u} = \big(u_N(x_0), u_N(x_1), \ldots, u_N(x_N)\big)^t, \quad \boldsymbol{f} = \big(g_-, f(x_1), \ldots, f(x_{N-1}), g_+\big)^t. \tag{2.31}$$

The important implication of (2.23) is that $\boldsymbol{Q}$ offers a nearly optimal preconditioner for the ill-conditioned system (2.30). More notably, it can precondition a general second-order BVP with variable coefficients. On the other hand, the "inverse" matrix $\boldsymbol{Q}$ can be computed in a stable and fast manner as to be shown in Sect. 3.

Note that under the Dirichlet boundary conditions (i.e., $\mathscr{B}_-[u] = u(-1)$ or $\mathscr{B}_+[u] = u(1)$), we augment the system in (2.30) by pretending $u(\pm 1)$ as unknown(s). As usual, we can move them to the right-handed side of (2.30), and then we can precondition the system by $\boldsymbol{Q}_{\rm in}$ defined in (2.21). Indeed, since $Q_j(\pm 1) = 0$ for $1 \le j \le N-1$, we obtain from (2.26) that

$$\boldsymbol{Q}_{\rm in}\big(\boldsymbol{D}_{\rm in}^{(2)} - \lambda^2 \boldsymbol{I}_{N-1}\big) \approx \boldsymbol{I}_{N-1}, \quad N \gg 1. \tag{2.32}$$

It is evident that a similar relation is available for the case involving the Dirichlet boundary condition at only one endpoint.

We depict in Fig. 1 the distribution of the largest and smallest eigenvalues of $\boldsymbol{QL}$ at the JGL points. We see that all their eigenvalues for various N are confined in $[\kappa_{\min}, \kappa_{\max}]$, which are concentrated around one for slightly large N. This agrees with (2.23).

2.3 Properties of (discrete) Eigenvalues

Consider the matrix eigenvalue problems:

$$\boldsymbol{Q}_{\rm in}\boldsymbol{v} = \kappa^{Q}\boldsymbol{v}, \quad \boldsymbol{Q}_{\rm in}^{(2)}\hat{\boldsymbol{v}} = \hat{\kappa}^{Q}\hat{\boldsymbol{v}}, \quad \boldsymbol{D}_{\rm in}^{(2)}\boldsymbol{u} = \kappa^{L}\boldsymbol{u}, \tag{2.33}$$

where $(\kappa^{Q}, \boldsymbol{v})$, $(\hat{\kappa}^{Q}, \hat{\boldsymbol{v}})$ and $(\kappa^{L}, \boldsymbol{u})$ are corresponding eigen-pairs. By (2.22) and (2.32),

$$\hat{\kappa}^{Q} - \lambda^2\kappa^{Q} = 1; \quad \kappa^{Q}(\kappa^{L} - \lambda^2) \approx 1, \quad N \gg 1. \tag{2.34}$$

It is known that the eigenvalues of $\boldsymbol{D}_{\rm in}^{(2)}$ are all real and distinct (cf. [35]), which implies the eigenvalues of $\boldsymbol{Q}_{\rm in}$ and $\boldsymbol{Q}_{\rm in}^{(2)}$ are all real and distinct when $N \gg 1$.

We next consider the model eigenvalue problem:

$$u''(x) = \kappa u(x), \quad x \in \Lambda; \quad u(\pm 1) = 0, \tag{2.35}$$

whose eigen-pairs are

$$\kappa_j = -\frac{j^2\pi^2}{4}, \quad u_j(x) = \begin{cases} \cos\left(\frac{1}{2} j\pi x\right), & j \text{ odd}, \\ \sin\left(\frac{1}{2} j\pi x\right), & j \text{ even}, \end{cases} \quad j = 1, 2, \ldots. \tag{2.36}$$

The Lagrange collocation (LCOL) scheme for (2.35) is given in (2.33) (third identity). Also, applying the collocation using the new basis $\{Q_j\}$ (with $\lambda > 0$) to (2.35), we obtain from (2.22) that

$$\left(\lambda^2 \boldsymbol{Q}_{\text{in}} + \boldsymbol{I}_{N-1}\right)\boldsymbol{v} = \kappa^B \boldsymbol{Q}_{\text{in}}\boldsymbol{v}, \quad \text{so} \quad \boldsymbol{Q}_{\text{in}}\boldsymbol{v} = \frac{1}{\kappa^B - \lambda^2}\boldsymbol{v}.$$

By (2.33)–(2.34), we can compute the approximate eigenvalues of (2.35) using new scheme by

$$\kappa^B = \frac{1}{\kappa^Q} + \lambda^2 = \frac{\hat{\kappa}^Q - 1}{\lambda^2}. \tag{2.37}$$

According to [35], about a portion of $2N/\pi$ out of N discrete eigenvalues of $\boldsymbol{D}_{\text{in}}^{(2)}$ can approximate the continuous eigenvalues of (2.35) with an accuracy of at least two or three digits. In Fig. 2, we plot the eigenvalues with $N = 50$ and $\lambda = 1$. We see that the two schemes (new scheme and LCOL) $\kappa^B \approx \kappa^L$. This shows the new basis has an approximability and resolutions similar to the polynomial basis. Figure 3 shows the spectral radius $\sigma(\boldsymbol{Q}_{\text{in}})$ of $\boldsymbol{Q}_{\text{in}}$, which has the behavior $\sigma(\boldsymbol{Q}_{\text{in}}) \approx 4/(4\lambda^2 + \pi^2)$ for $N \gg 1$. Indeed, by (2.34), we have

$$\sigma(\boldsymbol{Q}_{\text{in}}) = \max_j \left\{|\kappa_j^Q|\right\} \approx \frac{1}{|\kappa_1^L - \lambda^2|} \approx \frac{4}{4\lambda^2 + \pi^2}, \tag{2.38}$$

where we used $\kappa_1^L \approx -\pi^2/4$ in (2.36).

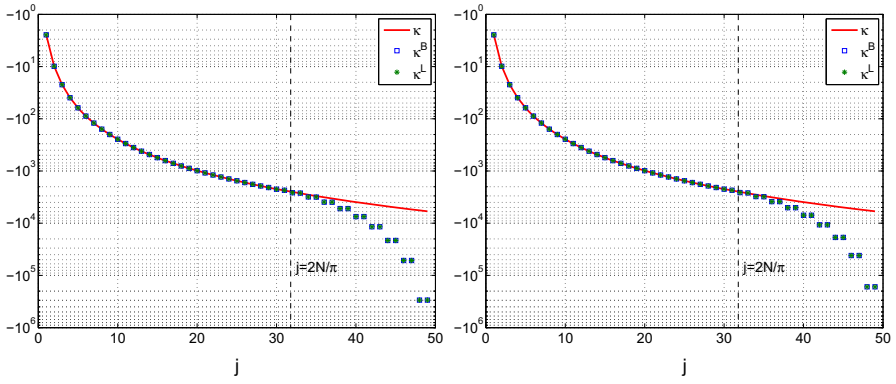

Fig. 2 Continuous and discrete eigenvalues with $N = 50$ and $\lambda = 1$. *Left* $\alpha = \beta = -1/2$. *Right* $\alpha = \beta = 0$

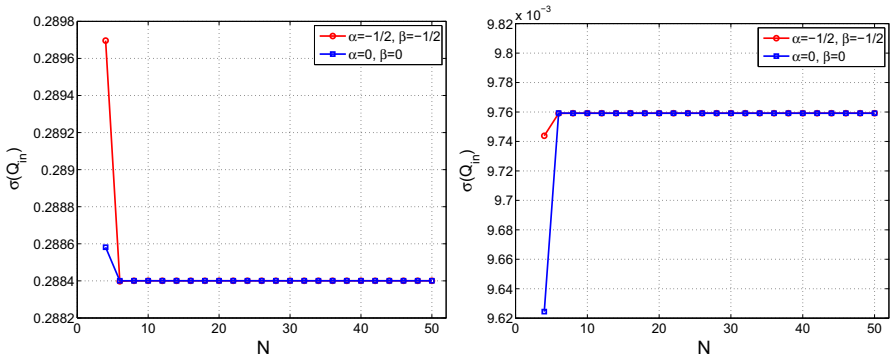

Fig. 3 Spectral radius of $\boldsymbol{Q}_{\text{in}}$. *Left* $\lambda = 1$. *Right* $\lambda = 10$

3 Rapid and Stable Computation of the New Basis

In this section, we provide a fast, stable and accurate means to compute the new basis $\{Q_j\}$ in Theorem 2.2. Note that it only suffices to evaluate $\mathcal{I}_\lambda^{\pm}[f](x_i)$ (defined in (2.4)) with $f = l_j$ (defined in (2.7)). The efficient algorithm is built upon the fast convolution scheme particularly for the exponential kernel (see, e.g., [1]), and recurrence formulas of Jacobi polynomials.

3.1 Fast Convolution Algorithm

We first compute $\mathcal{I}_\lambda^{\pm}[f]$ with $f(x) = P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial of degree n and with parameters $\alpha, \beta > -1$ (cf. [30]). Recall the property:

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \quad x \in (-1, 1). \tag{3.1}$$

With this, we can compute one integral from the other as follows.

Proposition 3.1 *Let $\mathcal{I}_\lambda^{\pm}$ be the integral operators defined in (2.4). Then we have*

$$\mathcal{I}_\lambda^{+}\big[P_n^{(\alpha,\beta)}\big](x) = (-1)^n\, \mathcal{I}_\lambda^{-}\big[P_n^{(\beta,\alpha)}\big](-x), \tag{3.2}$$

for $\alpha, \beta > -1$ and $n \geq 0$.

Proof The identity (3.2) can be verified directly from (2.4) to (3.1) and change of variables in integration. □

In view of this, we just focus on the computation of $\mathcal{I}_\lambda^{-}[f](x)$. Following e.g., [1], we have the important recurrence relation.

Proposition 3.2 *Let $\{x_i\}_{i=0}^N \subseteq [-1, 1]$ (with $x_0 = -1$) be any set of distinct points (e.g., JGL points) arranged in ascending order, and let $\mathcal{I}_\lambda^{\pm}[f](x)$ be the integrals defined in (2.4). Then for any $\lambda \neq 0$, we have*

$$\mathcal{I}_\lambda^{-}[f](x_i) = e^{-\lambda\Delta_i}\, \mathcal{I}_\lambda^{-}[f](x_{i-1}) + \frac{1}{2\lambda} \int_{x_{i-1}}^{x_i} e^{-\lambda(x_i - y)} f(y)\, \mathrm{d}y, \tag{3.3}$$

and $\mathcal{I}_\lambda^{-}[f](x_0) = 0$, where $\Delta_i = x_i - x_{i-1}$ for $1 \leq i \leq N$.

Proof By the definition (2.4) and a direct calculation, we have

$$\begin{aligned}\mathcal{I}_\lambda^-[f](x_i) &= \frac{1}{2\lambda}\int_{-1}^{x_i} e^{-\lambda(x_i-y)} f(y)\,\mathrm{d}y \\ &= \frac{1}{2\lambda}\int_{-1}^{x_{i-1}} e^{-\lambda(x_{i-1}+\Delta_i-y)} f(y)\,\mathrm{d}y + \frac{1}{2\lambda}\int_{x_{i-1}}^{x_i} e^{-\lambda(x_i-y)} f(y)\,\mathrm{d}y \\ &= e^{-\lambda\Delta_i}\,\mathcal{I}_\lambda^-[f](x_{i-1}) + \frac{1}{2\lambda}\int_{x_{i-1}}^{x_i} e^{-\lambda(x_i-y)} f(y)\,\mathrm{d}y.\end{aligned}$$

This ends the proof. □

Thanks to the exponential convolution kernel, we are able to compute the integral values $\{\mathcal{I}_\lambda^-[f](x_i)\}$ recursively, which only require the evaluation of the integrals at the local subintervals $\{[x_{i-1}, x_i]\}_{i=1}^N$. This overcomes the burden of history dependence of integration.

3.2 Useful Formulas

With (3.3) at our disposal, it suffices to compute the local integrals. More specifically, for the new basis $\{Q_j\}$ in Theorem 2.1, we only need to compute

$$S_j^i := \mathcal{I}_\lambda^-[l_j](x_i) = e^{-\lambda\Delta_i}\,\mathcal{I}_\lambda^-[l_j](x_{i-1}) + \frac{1}{2\lambda}\int_{x_{i-1}}^{x_i} e^{-\lambda(x_i-y)} l_j(y)\,\mathrm{d}y, \tag{3.4}$$

for $1 \le i \le N$, $1 \le j \le N-1$. Note that to better represent the recurrence relations, we use the notation S_j^i, μ_j^n, Z_n^i etc., where the superscript (resp. subscript) is the row (resp. column) index when we assemble the matrices.

The Lagrange interpolating basis polynomials $\{l_j\}$ are defined in (2.7), which can be represented in terms of Jacobi polynomials.

Proposition 3.3 *Let $\{x_j, \omega_j\}$ be the JGL points and quadrature weights. Then l_j defined in (2.7) has the representation:*

$$l_j(x) = \sum_{n=0}^{N-2} \mu_j^n\, P_n^{(\alpha,\beta)}(x), \quad 1 \le j \le N-1, \tag{3.5}$$

where

$$\begin{aligned}\mu_j^n &:= \frac{1}{\gamma_n^{(\alpha,\beta)}}\left\{P_n^{(\alpha,\beta)}(-1) l_j(-1)\omega_0 + P_n^{(\alpha,\beta)}(x_j)\omega_j + P_n^{(\alpha,\beta)}(1) l_j(1)\omega_N\right\}, \\ l_j(-1) &= -\frac{1-x_j}{2(\beta+1)}\frac{P_N^{(\alpha,\beta)}(-1)}{P_N^{(\alpha,\beta)}(x_j)}, \quad l_j(1) = -\frac{1+x_j}{2(\alpha+1)}\frac{P_N^{(\alpha,\beta)}(1)}{P_N^{(\alpha,\beta)}(x_j)}.\end{aligned} \tag{3.6}$$

To avoid distraction from the main results, we provide the derivation of the above formulas in Appendix 8.

With the aid of (3.5), the evaluation of (3.4) boils down to computing

$$Z_n^i := \int_{x_{i-1}}^{x_i} e^{-\lambda(x_i-y)} P_n^{(\alpha,\beta)}(y)\,\mathrm{d}y, \tag{3.7}$$

for $1 \le i \le N$ and $0 \le n \le N-2$.

The following recurrence relation is essential for the fast computation of the basis functions.

Proposition 3.4 *For $\alpha, \beta > -1$, let $\{x_i\}_{i=0}^N$ be the JGL points arranged in ascending order. Then we have that for $1 \le i \le N$,*

$$\lambda a_n Z_{n-1}^i + (1 + \lambda b_n) Z_n^i + \lambda c_n Z_{n+1}^i = g_n^i, \quad n \ge 1; \quad Z_0^i = (1 - e^{-\lambda \Delta_i})/\lambda, \tag{3.8}$$

where $\Delta_i = x_i - x_{i-1}$, $\{a_n, b_n, c_n\}$ are defined in (B.8), and

$$\begin{aligned} g_n^i =& a_n P_{n-1}^{(\alpha,\beta)}(x_i) + b_n P_n^{(\alpha,\beta)}(x_i) + c_n P_{n+1}^{(\alpha,\beta)}(x_i) \\ & - e^{-\lambda \Delta_i} \left\{ a_n P_{n-1}^{(\alpha,\beta)}(x_{i-1}) + b_n P_n^{(\alpha,\beta)}(x_{i-1}) + c_n P_{n+1}^{(\alpha,\beta)}(x_{i-1}) \right\}. \end{aligned} \tag{3.9}$$

Proof Inserting the formula (7.7) into (3.7), we obtain (3.8)–(3.9) straightforwardly by using integration by parts. Note that Z_0^i is computed directly from (3.7). □

Remark 3.1 Observe from (B.8) that for fixed $\alpha, \beta > -1$,

$$a_n = O(n^{-1}), \quad b_n = O(n^{-2}), \quad c_n = O(n^{-1}).$$

The naive updating Z_{n+1}^i from Z_{n-1}^i and Z_n^i recursively by using (3.8) with the initial data Z_0^i and Z_1^i, appears unstable, as the coefficient $(1 + \lambda b_n)/\lambda c_n$ grows like $O(n)$.

Instead, we treat (3.8) as a tridiagonal system equipped with the boundary values Z_0^i and Z_{N-2}^i (to be pre-computed). Inspect that for large n, the system is strictly diagonally dominant. More importantly, the coefficients are independent of i, so we can pre-compute and store e.g., the LU-decomposition of the tridiagonal coefficient matrix, and then update $\{g_n^i\}$ for each i. This allows us to evaluate all $\{Z_n^i\}$ in a fast, stable and accurate manner at a cost comparable to the use of a recurrence relation. □

3.3 Summary of the Algorithm

We now have all pieces of the puzzle ready, and summarize the algorithm for computing $\left\{\mathcal{I}_\lambda^-[l_j](x_i)\right\}$ of the new basis functions $\{Q_j\}$ in Theorem 2.1, i.e., the matrix $\boldsymbol{Q}$ in (2.21) as follows.

Algorithm for computing $\{S_j^i\}$ $(0 \le i \le N, 1 \le j \le N-1)$ in (3.4)

1. Initialization:
   - (i) Pre-compute the JGL points and weights $\{x_j, \omega_j\}_{j=0}^N$, and Jacobi polynomials at JGL points $\left\{P_n^{(\alpha,\beta)}(x_j)\right\}_{0 \le j \le N}^{0 \le n \le N-2}$.
   - (ii) Store the LU-decomposition of the tridiagonal matrix $\boldsymbol{A}$ with main diagonal $\{1 + \lambda b_n\}_{n=1}^{N-3}$ and lower (resp. upper) off-diagonal $\{\lambda a_n\}_{n=2}^{N-3}$ (resp. $\{\lambda c_n\}_{n=1}^{N-4}$).
2. For $j = 1, \ldots, N-1$,
   - a) From (3.6), we compute
   $$\boldsymbol{\mu}_j = \left(\mu_j^0, \mu_j^1, \ldots, \mu_j^{N-2}\right)^t.$$
   - b) For $i = 1, \ldots, N$,
     - (i) Solve the tridiagonal system by substitution
     $$\begin{cases} \boldsymbol{A}\boldsymbol{Z}^i = \boldsymbol{g}^i \quad \text{with} \quad \boldsymbol{Z}^i = \left(Z_1^i, \ldots, Z_{N-3}^i\right)^t, \quad \text{and} \\ \boldsymbol{g}^i = \left(g_1^i - \lambda a_1 Z_0^i, g_2^i, \ldots, g_{N-4}^i, g_{N-3}^i - \lambda c_{N-3} Z_{N-2}^i\right)^t, \end{cases} \tag{3.10}$$

where Z^i_{N-2} in (3.7) can be computed by the Legendre-Gauss quadrature.

(ii) From (3.3), (3.5) and (3.7), we compute

$$S^i_j = e^{-\lambda\Delta_i} S^{i-1}_j + \frac{1}{2\lambda}\, \boldsymbol{Z}^i \cdot \boldsymbol{\mu}_j; \quad S^0_j = 0. \tag{3.11}$$

Endfor of i.

Endfor of j.

Remark 3.2 If $|\lambda| \gg 1$, we find it is necessary to re-normalize the basis as $\{\tilde{Q}_j := \lambda^2 Q_j\}_{j=1}^{N-1}$. Accordingly, we use $\lambda^2 l_j$ in place of l_j in (3.4), reset (3.7) as

$$Z^i_n := \int_{x_{i-1}}^{x_i} \lambda e^{-\lambda(x_i - y)} P_n^{(\alpha,\beta)}(y)\, \mathrm{d}y, \tag{3.10}$$

and modify the above algorithm. A direct calculation from the mean value theorem leads to

$$|Z^i_n| \le \max_{x\in[-1,1]} \{|P_n^{(\alpha,\beta)}(x)|\}\big(1 - e^{-\lambda\Delta_i}\big). \tag{3.11}$$

Also observe from (2.16) that the summation in the evaluation of the basis becomes

$$\sum_{j=1}^{N-1} \mathscr{L}_\lambda[u](x_j) Q_j(x) = \sum_{j=1}^{N-1} \frac{\mathscr{L}_\lambda[u](x_j)}{\lambda^2} \tilde{Q}_j(x), \tag{3.12}$$

where for $u \in C^2(\Lambda)$, we have

$$\lim_{|\lambda|\to\infty} \frac{\mathscr{L}_\lambda[u](x_j)}{\lambda^2} = -u(x_j).$$

In view of these, we see that the computation becomes stable, and one should scale λ^2, even for moderate large $|\lambda|$. □

In Fig. 4, we plot the first five basis functions.

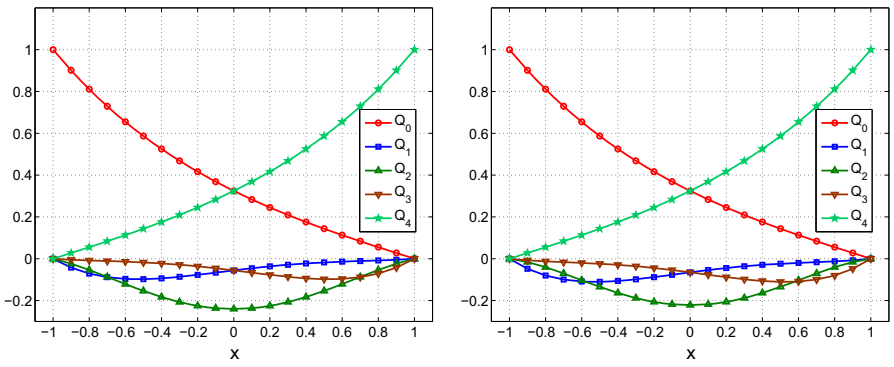

Fig. 4 Plots of $\{Q_j\}_{j=0}^4$ with $\lambda = 1$, $\mathscr{B}_\pm[u] = u(\pm 1)$ and $N = 4$. *Left* $\alpha = \beta = -1/2$. *Right* $\alpha = \beta = 0$

4 Interpolation Error Estimates

Hereafter, we study the approximability of the new basis. To fix the idea, we restrict our attentions to the Birkhoff interpolation (2.16) with $\mathscr{B}_{\pm}[u] = u(\pm 1)$, and focus on the analysis of interpolation at Legendre–Gauss–Lobatto (LGL) and Chebyshev–Gauss–Lobatto (CGL) points.

We first introduce some notation. Given a generic weight function ϖ on Λ, let $L^2_\varpi(\Lambda)$ be the space of all square integrable functions on Λ, with the inner product and norm denoted by $(\cdot,\cdot)_\varpi$ and $\|\cdot\|_\varpi$, respectively, as usual. Let $H^m_\varpi(\Lambda)$ for $m \geq 1$ be the usual weighted Sobolev space with the norm and semi-norm, respectively, defined by

$$\|u\|_{m,\varpi} = \Big(\sum_{k=0}^{m} \|u^{(k)}\|^2_\varpi\Big)^{1/2}, \quad |u|_{m,\varpi} = \|u^{(m)}\|_\varpi.$$

In what follows, for any $u \in C^2(\Lambda)$, we denote

$$\begin{aligned} f(x) &= \mathscr{L}_\lambda[u](x) = u''(x) - \lambda^2 u(x), \\ e_N(x) &= \big(u - \mathbb{I}^B_N u\big)(x), \quad \tilde{e}^f_N(x) = \big(f - \mathbb{I}^G_{N-2} f\big)(x), \end{aligned} \tag{4.1}$$

where $\mathbb{I}^B_N$ is defined in (2.16) with $\mathscr{B}_{\pm}[u] = u(\pm 1)$, and $\mathbb{I}^G_{N-2}$ is the Lagrange interpolation operator at the interior JGL points $\{x_i\}_{i=1}^{N-1}$ (i.e., the Jacobi–Gauss interpolation operator at zeros of $P^{(\alpha+1,\beta+1)}_{N-1}(x)$). It is important to notice the following error equation:

$$\mathscr{L}_\lambda[e_N](x) = e''_N(x) - \lambda^2 e_N(x) = \tilde{e}^f_N(x), \quad x \in \Lambda; \quad e_N(\pm 1) = 0. \tag{4.2}$$

Indeed, by (2.15),

$$\big(\mathbb{I}^B_N u\big)''(x) - \lambda^2 \big(\mathbb{I}^B_N u\big)(x) = \big(\mathbb{I}^G_{N-2} f\big)(x), \tag{4.3}$$

so (4.2) follows from subtracting (4.3) from the first equation in (4.1) immediately. Introduce the non-uniformly Jacobi-weighted Sobolev space:

$$B^m_{\alpha,\beta}(\Lambda) = \big\{u : u^{(k)} \in L^2_{\omega^{(\alpha+k,\beta+k)}}(\Lambda),\ 0 \leq k \leq m\big\}, \quad m \in \mathbb{N}.$$

Recall the Jacobi–Gauss interpolation error estimate [27, Theorem. 3.41]: For $\alpha, \beta > -1$, and any $f \in B^m_{\alpha+1,\beta+1}(\Lambda)$ with $1 \leq m \leq N+1$,

$$\|f - \mathbb{I}^G_{N-2} f\|_{\omega^{(\alpha+1,\beta+1)}} \leq cN^{-m} \|f^{(m)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}}, \tag{4.4}$$

where c is a positive constant independent of N and f. This, together with (4.2), immediately implies

$$\|\mathscr{L}_\lambda[e_N]\|_{\omega^{(\alpha+1,\beta+1)}} \leq cN^{-m} \|f^{(m)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}}, \tag{4.5}$$

for $m \geq 1$, $\alpha, \beta > -1$, and $\lambda \neq 0$, where $f = \mathscr{L}_\lambda[u]$.

In fact, we can derive the H^2-estimates in the following sense.

Theorem 4.1 *For $-1 < \alpha, \beta \leq 0$, and for real $\lambda \neq 0$, if $u'' \in L^2_{\omega^{(\alpha+m+1,\beta+m+1)}}(\Lambda)$ and $u \in B^m_{\alpha,\beta}(\Lambda)$ with $m \geq 1$, then we have*

$$\|u - \mathbb{I}^B_N u\|_{2,\omega^{(\alpha+1,\beta+1)}} \leq cN^{-m}\big(\|u^{(m+2)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}} + \lambda^2 \|u^{(m)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}}\big), \tag{4.6}$$

where c is a positive constant independent of u and N.

Proof Multiplying (4.2) by $e_N\omega^{(\alpha+1,\beta+1)}$, and integrating the resulted equation over $(-1, 1)$, we obtain from integration by parts that

$$\begin{aligned}\|e_N'\|^2_{\omega^{(\alpha+1,\beta+1)}} + \lambda^2\|e_N\|^2_{\omega^{(\alpha+1,\beta+1)}} + \frac{1}{2}\int_{-1}^{1} e_N^2(x)W(x)\omega^{(\alpha-1,\beta-1)}(x)dx \\ = -(\tilde{e}_N^f, e_N)_{\omega^{(\alpha+1,\beta+1)}} \le \frac{\lambda^2}{2}\|e_N\|^2_{\omega^{(\alpha+1,\beta+1)}} + \frac{1}{2\lambda^2}\|\tilde{e}_N^f\|^2_{\omega^{(\alpha+1,\beta+1)}},\end{aligned} \tag{4.7}$$

where by a direct calculation,

$$W(x) = -(\alpha+\beta+2)(\alpha+\beta+1)x^2 - 2(\alpha-\beta)(\alpha+\beta+1)x + \alpha+\beta+2-(\alpha-\beta)^2. \tag{4.8}$$

From [27, P. 126], we have $W(x) \ge 0$ if $-1 < \alpha, \beta \le 0$. Then (4.7) implies

$$\|e_N'\|^2_{\omega^{(\alpha+1,\beta+1)}} + \frac{\lambda^2}{2}\|e_N\|^2_{\omega^{(\alpha+1,\beta+1)}} \le \frac{1}{2\lambda^2}\|\tilde{e}_N^f\|^2_{\omega^{(\alpha+1,\beta+1)}}. \tag{4.9}$$

Thus, by (4.5),

$$\|e_N''\|_{\omega^{(\alpha+1,\beta+1)}} \le \lambda^2\|e_N\|_{\omega^{(\alpha+1,\beta+1)}} + cN^{-m}\|f^{(m)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}}. \tag{4.10}$$

Noting that

$$\|f^{(m)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}} \le c(\|u^{(m+2)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}} + \lambda^2\|u^{(m)}\|_{\omega^{(\alpha+m+1,\beta+m+1)}}), \tag{4.11}$$

we obtain (4.6) from the above three estimates. □

Remark 4.1 For the Birkhoff interpolation at LGL points (i.e., $\alpha = \beta = 0$), it is clear that $W(x) = 2(1 - x^2)$ in (4.8), so the interpolation error in L^2-norm $\|e_N\|$ can be bounded by the same upper bound in (4.6).

In the Chebyshev case (i.e., $\alpha = \beta = -1/2$), we have $W(x) = 1$, so the estimate (4.6) is also valid for $\|e_N\|_{\omega^{(-3/2,-3/2)}}$. □

Remark 4.2 Theorem 4.1 indicates that the order of convergence in weighted H^2-norm is as optimal as usual polynomial interpolation. However, it appears nontrivial to rigorously show the optimal order of convergence in H^1-norm and L^2-norm. In fact, we can apply integration by parts to $-(\tilde{e}_N^f, e_N)_{\omega^{(\alpha+1,\beta+1)}}$ in (4.7), and shift the derivative to e_N.

- For $-1 < \alpha, \beta < 0$, we have

$$-(\tilde{e}_N^f, e_N)_{\omega^{(\alpha+1,\beta+1)}} = (\partial_x^{-1}\tilde{e}_N^f, e_N')_{\omega^{(\alpha+1,\beta+1)}} + (\partial_x^{-1}\tilde{e}_N^f, qe_N)_{\omega^{(\alpha,\beta)}}, \tag{4.12}$$

  where $\partial_x^{-1}\tilde{e}_N^f(x) = \int_{-1}^{x}\tilde{e}_N^f(y)dy$, and $q = (\beta-\alpha)-(\alpha+\beta+2)x$. Recall the inequality (cf. [27, (B.40)]):

$$\|e_N\|_{\omega^{(\alpha-1,\beta-1)}} \le c\|e_N'\|_{\omega^{(\alpha+1,\beta+1)}}. \tag{4.13}$$

  Thus, (4.9) can be improved into

$$\|e_N'\|^2_{\omega^{(\alpha+1,\beta+1)}} + \lambda^2\|e_N\|^2_{\omega^{(\alpha+1,\beta+1)}} \le c\|\partial_x^{-1}\tilde{e}_N^f\|^2_{\omega^{(\alpha+1,\beta+1)}}. \tag{4.14}$$

- For $\alpha = \beta = 0$, we apply the same technique and obtain from (4.7) that for some $0 < s < 1$,

$$\begin{aligned}\|e_N'\|^2_{\omega^{(1,1)}} + \lambda^2\|e_N\|^2_{\omega^{(1,1)}} + \|e_N\|^2 \le |(\partial_x^{-1}\tilde{e}_N^f, e_N')_{\omega^{(1,1)}}| + 2|(\partial_x^{-1}\tilde{e}_N^f, xe_N)| \\ \le \|\partial_x^{-1}\tilde{e}_N^f\|_{\omega^{(1,1)}}\|e_N'\|_{\omega^{(1,1)}} + 2\|\partial_x^{-1}\tilde{e}_N^f\|_{\omega^{(s,s)}}\|xe_N\|_{\omega^{(-s,-s)}}.\end{aligned} \tag{4.15}$$

Table 1 Order of convergence in $\|\tilde{e}_N^f\|_\omega$ and $\|\partial_x^{-1}\tilde{e}_N^f\|_\omega$ for the Chebyshev and Legendre cases

N	Chebyshev ($\omega=\omega^{(1/2,1/2)}$)				Legendre ($\omega=\omega^{(0.9,0.9)}$)			
	$\|\tilde{e}_N^f\|_\omega$	Order	$\|\partial_x^{-1}\tilde{e}_N^f\|_\omega$	Order	$\|\tilde{e}_N^f\|_\omega$	Order	$\|\partial_x^{-1}\tilde{e}_N^f\|_\omega$	Order
16	3.54e−04	–	2.43e−05	–	3.35e−04	–	2.25e−05	–
24	1.25e−04	2.57	5.82e−06	3.53	1.20e−04	2.53	5.47e−06	3.49
32	6.02e−05	2.53	2.12e−06	3.51	5.85e−05	2.50	2.02e−06	3.47
40	3.43e−05	2.52	9.69e−07	3.51	3.35e−05	2.50	9.31e−07	3.47
48	2.17e−05	2.51	5.12e−07	3.50	2.13e−05	2.49	4.95e−07	3.47
56	1.47e−05	2.51	2.98e−07	3.50	1.45e−05	2.49	2.90e−07	3.47
64	1.05e−05	2.51	1.87e−07	3.50	1.04e−05	2.49	1.82e−07	3.48
72	7.85e−06	2.50	1.24e−07	3.50	7.75e−06	2.49	1.21e−07	3.48

We proceed by recalling the inequality (cf. [27, (B.42)]): for $a, b > -1$ and $v \in H^1_{\omega^{(a+2,b+2)}}(\Lambda)$ with $v(x_0)=0$ for some $x_0 \in \Lambda$, we have

$$\|v\|_{\omega^{(a,b)}} \le c\|v'\|_{\omega^{(a+2,b+2)}}. \tag{4.16}$$

This implies

$$\begin{aligned}\|xe_N\|_{\omega^{(-s,-s)}} &\le c\|(xe_N)'\|_{\omega^{(2-s,2-s)}} \le c\big(\|e_N'\|_{\omega^{(2-s,2-s)}} + \|e_N\|_{\omega^{(2-s,2-s)}}\big)\\ &\le c\big(\|e_N'\|_{\omega^{(1,1)}} + \|e_N\|\big).\end{aligned} \tag{4.17}$$

Then using the Cauchy–Schwarz inequality, we obtain from (4.15) and (4.17) that

$$\|e_N'\|^2_{\omega^{(1,1)}} + \lambda^2\|e_N\|^2_{\omega^{(1,1)}} + \|e_N\|^2 \le c\|\partial_x^{-1}\tilde{e}_N^f\|^2_{\omega^{(s,s)}}, \quad 0<s<1. \tag{4.18}$$

We expect to have $\|\partial_x^{-1}\tilde{e}_N^f\|_\omega \sim N^{-1}\|\tilde{e}_N^f\|_\omega$ with $\omega=\omega^{(\alpha+1,\beta+1)}, \omega^{(s,s)}$, respectively, for two cases. However, it seems open to prove this rigorously, and the major difficulty lies in the estimates of polynomial interpolation errors in negative Sobolev norms. Here, we just provide a numerical example to illustrate the gain in order. For this purpose, we choose f with limited regularity as follows

$$f(x)=\begin{cases}\dfrac{x^2}{2}+x-1, & x\le 0,\\ x-1, & x>0.\end{cases} \tag{4.19}$$

We tabulate in Table 1 the (discrete) weighted L^2-errors in different sense. It is evident that with the above treatment, the convergence rate is one order higher as expected. □

5 New Collocation Methods

In this section, we construct new collocation schemes based upon the new basis $\{Q_j\}$ for various PDEs including steady-state and time-dependent problems.

5.1 Boundary Value Problems

Consider

$$u''(x) + r(x)u'(x) + s(x)u(x) = f(x), \quad x \in \Lambda, \tag{5.1}$$

where the given functions $r, s, f \in C(\Lambda)$. To fix the idea, we consider (5.1) with Dirichlet boundary conditions: $u(\pm 1) = u_\pm$.

Let $\{x_i\}_{i=0}^N$ be the JGL points as before. Then the collocation scheme for (5.1) is to find $u_N \in \mathcal{P}_N$ such that

$$u_N''(x_i) + r(x_i)u_N'(x_i) + s(x_i)u_N(x_i) = f(x_i), \quad 1 \le i \le N-1; \quad u_N(\pm 1) = u_\pm. \tag{5.2}$$

(i) *Usual collocation scheme using Lagrange polynomial basis*. Let $\{h_j\}_{j=0}^N$ be the Lagrange basis polynomials defined in (2.19). We expand the numerical solution of (5.1) as

$$u_N(x) = u_- h_0(x) + \sum_{j=1}^{N-1} u_N(x_j)h_j(x) + u_+ h_N(x),$$

and substitute it into (5.2), leading to

$$\big(\boldsymbol{D}_{\rm in}^{(2)} + \boldsymbol{\Lambda}_r \boldsymbol{D}_{\rm in}^{(1)} + \boldsymbol{\Lambda}_s\big)\,\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{u}_B, \tag{5.3}$$

where

$$\boldsymbol{\Lambda}_r = \mathrm{diag}(r(x_1), r(x_2), \ldots, r(x_{N-1})), \quad \boldsymbol{\Lambda}_s = \mathrm{diag}(s(x_1), s(x_2), \ldots, s(x_{N-1})),$$
$$\boldsymbol{u} = (u_N(x_1), u_N(x_2), \ldots, u_N(x_{N-1}))^t, \quad \boldsymbol{f} = (f(x_1), f(x_2), \ldots, f(x_{N-1}))^t,$$
$$\boldsymbol{u}_B \text{ is the vector of } \big\{u_-(d_{i0}^{(2)} + r(x_i)d_{i0}^{(1)}) + u_+(d_{iN}^{(2)} + r(x_i)d_{iN}^{(1)})\big\}_{i=1}^{N-1}.$$

(ii) *New collocation scheme using $\{Q_j\}$ as basis functions*. Write

$$u_N(x) = u_- Q_0(x) + \sum_{j=1}^{N-1} v_j Q_j(x) + u_+ Q_N(x), \tag{5.4}$$

where $v_j = u_N''(x_j) - \lambda^2 u_N(x_j)$. Then the matrix form of (5.2) reads

$$\big(\boldsymbol{I}_{N-1} + \boldsymbol{\Lambda}_r \boldsymbol{Q}_{\rm in}^{(1)} + (\boldsymbol{\Lambda}_s + \lambda^2 \boldsymbol{I}_{N-1})\boldsymbol{Q}_{\rm in}\big)\boldsymbol{v} = \boldsymbol{f} - u_-\boldsymbol{v}_- - u_+\boldsymbol{v}_+, \tag{5.5}$$

here $\boldsymbol{v}_-$ and $\boldsymbol{v}_+$ are vectors with entries $\big\{r(x_i)q_{i0}^{(1)} + (s(x_i) + \lambda^2)q_{i0}\big\}_{i=1}^{N-1}$ and $\big\{r(x_i)q_{iN}^{(1)} + (s(x_i) + \lambda^2)q_{iN}\big\}_{i=1}^{N-1}$, respectively.

Remark 5.1 It is clear that if $r = 0$ and $s = -\lambda^2$, the numerical solution in (5.4) can be directly obtained without solving a linear system. □

In what follows, we take $r(x) = -(1 + \sin x)$, $s(x) = -e^x$ and test two exact solutions. One is sufficiently smooth:

$$u(x) = e^{(x^2-1)/2}, \quad x \in \Lambda, \tag{5.6}$$

and the other has a limited regularity:

Table 2 Errors, number of iterations and condition numbers for (5.1) with $\alpha = \beta = 0$, $\lambda = 1$ and the exact solution (5.6)

N	LCOL (5.3)			BCOL [32]			New scheme (5.5)		
	Conds	Iters	Errors	Conds	Iters	Errors	Conds	Iters	Errors
4	6.18e+00	3	1.28e−03	1.41	3	1.28e−03	1.40	3	1.08e−03
6	1.92e+01	6	2.20e−05	1.77	5	2.20e−05	1.63	5	1.88e−05
8	4.87e+01	9	3.85e−07	1.97	7	3.85e−07	1.72	7	3.41e−07
10	1.07e+02	12	6.06e−09	2.09	9	6.06e−09	1.79	8	5.49e−09
12	2.10e+02	15	8.67e−11	2.17	9	8.67e−11	1.84	9	7.98e−11
14	3.74e+02	20	1.12e−12	2.23	9	1.13e−12	1.88	9	1.05e−12
16	6.20e+02	25	3.80e−14	2.27	9	1.34e−14	1.91	9	1.24e−14

Table 3 Errors, number of iterations and condition numbers for (5.1) with $\alpha = \beta = 0$, $\lambda = 1$ and the exact solution (5.7)

N	LCOL (5.3)			BCOL [32]			New scheme (5.5)		
	Conds	Iters	Errors	Conds	Iters	Errors	Conds	Iters	Errors
4	6.18e+00	3	7.26e−03	1.41	3	7.26e−03	1.40	3	5.92e−03
8	4.87e+01	9	1.40e−04	1.97	7	1.40e−04	1.72	7	1.40e−04
16	6.20e+02	25	1.23e−05	2.27	9	1.23e−05	1.91	9	1.23e−05
32	9.13e+03	81	9.27e−07	2.42	9	9.27e−07	2.03	9	9.27e−07
64	1.41e+05	308	6.39e−08	2.50	9	6.39e−08	2.09	9	6.39e−08
128	2.21e+06	1319	5.13e−09	2.54	9	4.20e−09	2.12	9	4.20e−09
256	3.52e+07	7472	9.24e−09	2.56	9	2.69e−10	2.13	9	2.69e−10
512	5.60e+08	9982	3.88e+00	2.57	9	1.71e−11	2.14	9	1.71e−11
1024	8.95e+09	9814	5.37e+00	2.57	9	1.07e−12	2.14	9	1.07e−12
2048	1.43e+11	9998	5.87e+00	2.57	9	6.75e−14	2.15	9	6.17e−14

$$u(x) = \begin{cases} e^{\frac{x^2}{2}+1} + e^{\frac{x^2}{2}}, & -1 \le x < 0, \\ e^{\frac{x^2}{2}+1} + \frac{x^2}{2} + 1, & 0 \le x \le 1, \end{cases} \quad x \in \Lambda, \tag{5.7}$$

which is in $C^3(\Lambda)$.

In Tables 2 and 3, we compare the condition numbers, number of iterations (using BiCGSTAB in Matlab with options tol=10^{-14} and maxit=10,000) and errors (L^∞-norm) of the three schemes: LCOL (5.3), the Birkhoff collocation (BCOL) in [32], and the new scheme (5.5) for (5.1). As expected, the usual collocation scheme using Lagrange interpolating polynomial basis suffers from severe round-off errors, and the iterative method fails to converge for large N, due to the ill-conditioning of the linear system. In contrast, we see that the other two collocation schemes do not have such a deficiency. They are consistently stable, and enjoy the expected order of convergence within a few iterations even for a large number of collocation points. Compared with BCOL in [32], the new approach has advantage for constant coefficient problems (cf. Remark 5.1). The results also show that the new non-polynomial approach as accurate as the polynomial-based approach in [32].

As a second example, we consider the collocation scheme for the Bessel-type equation:

$$\begin{aligned} &-\frac{1}{r}\frac{d}{dr}\Big(r\frac{dv}{dr}\Big)+\frac{l^2}{r^2}v-k^2v=0, \quad r\in(a,b), \quad a>0;\\ &v(a)=g_1, \quad v'(b)-\mathrm{i}kv(b)=g_2, \end{aligned} \tag{5.8}$$

for integer l, where $k \gg 1$ is the wavenumber, and g_1, g_2 are given data. It is known that the two-dimensional time-harmonic acoustic wave equation in an annular domain: $r \in (a, b), \theta \in [0, 2\pi)$ reduces to (5.8) if one uses Fourier expansion in θ-direction (see, e.g., [29,31]).

To facilitate the use of a collocation scheme, we make a change of variable to remove the first-order derivative (cf. [31]), that is,

$$r=a+\frac{x+1}{2}(b-a), \quad u(x)=\sqrt{r}v(r), \quad r\in(a,b), \quad x\in(-1,1). \tag{5.9}$$

Then, we can convert (5.8) into

$$\begin{aligned} &u''(x)+\hat{k}^2u(x)-s(x)u(x)=0, \quad x\in(-1,1),\\ &u(-1)=g_-, \quad u'(1)-\eta\,u(1)=g_+, \end{aligned} \tag{5.10}$$

where

$$\hat{k}=\frac{b-a}{2}k, \quad s(x)=\Big(\frac{b-a}{2}\Big)^2\frac{4l^2-1}{4r^2}, \quad g_-=\sqrt{a}\,g_1, \quad g_+=\frac{b-a}{2}\sqrt{b}\,g_2,$$
$$\eta=\mathrm{i}\hat{k}+\frac{b-a}{4b}.$$

Here, we use the new basis $\{Q_j\}$ with $\lambda = \mathrm{i}\hat{k}$, and look for the numerical solution

$$u_N(x)=g_-Q_0(x)+\sum_{j=1}^{N-1}v_jQ_j(x)+g_+Q_N(x)\in\mathcal{Q}_N, \tag{5.11}$$

such that

$$\begin{aligned} &u_N''(x_i)+\hat{k}^2u_N(x_i)-s(x_i)u_N(x_i)=0, \quad 1\le i\le N-1,\\ &u_N(-1)=g_-, \quad u_N'(1)-\eta\,u_N(1)=g_+. \end{aligned} \tag{5.12}$$

Thanks to (2.22), the linear system of (5.12) reads

$$\big(\boldsymbol{I}_{N-1}-\boldsymbol{\Lambda}_s\,\boldsymbol{Q}_{\mathrm{in}}\big)\,\boldsymbol{v}=\boldsymbol{f}, \tag{5.13}$$

where

$$\boldsymbol{\Lambda}_s=\mathrm{diag}(s(x_1),s(x_2),\ldots,s(x_{N-1})), \quad \boldsymbol{v}=(v_1,v_2,\ldots,v_{N-1})^t,$$

and $\boldsymbol{f}$ is a vector of $\big\{s(x_i)(q_{i0}g_-+q_{iN}g_+)\big\}_{i=1}^{N-1}$.

We test the exact solution $v(r) = J_1(kr)/k$ of (5.8) (with $l = 1$, $a = 1$ and $b = 2$), where $J_1(\cdot)$ is the first kind Bessel function of order 1. It is highly oscillatory when $k \gg 1$. In Fig. 5, we depict the maximum pointwise errors against N for different wavenumber k. We see that the errors decay exponentially when $N > k(b-a)/2$, and the collocation solver is very stable for high wavenumber k. Indeed, the conditioning of the linear system (5.13) is independent of N and k (see Table 4). It is noteworthy that the well-designed spectral-Galerkin solver for (5.8) still has a condition number of $O(k^2)$.

Remark 5.2 The aforementioned approach works for $a > 0$, while for $a = 0$, we have to impose a suitable "pole" condition at the origin. Then we can follow the treatment in [19] to construct the collocation scheme. □

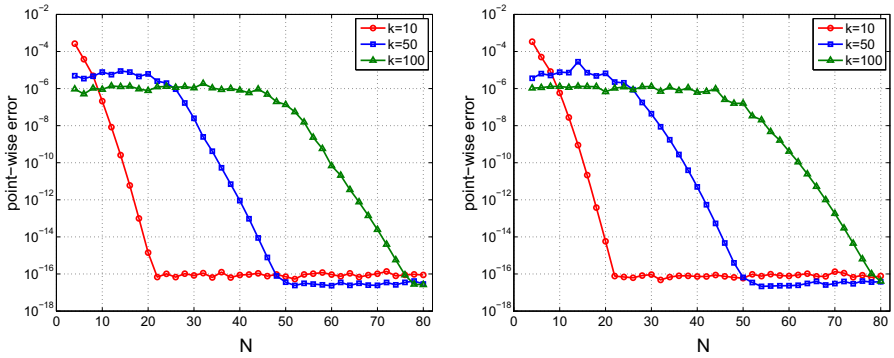

Fig. 5 Maximum pointwise errors vs N for (5.8). *Left* $\alpha = \beta = -1/2$. *Right* $\alpha = \beta = 0$

Table 4 Condition number of $\boldsymbol{I}_{N-1} - \boldsymbol{\Lambda}_s \boldsymbol{Q}_{\text{in}}$ with $\alpha = \beta = 0$ in (5.13)

k	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	k	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
50	1.006140	1.006138	1.006137	1.006136	500	1.000165	1.000640	1.000643	1.000643
100	1.003131	1.003130	1.003129	1.003129	800	1.000069	1.000134	1.000403	1.000403
200	1.001588	1.001587	1.001587	1.001587	1000	1.000074	1.000068	1.000323	1.000323

5.2 Time-Dependent Problems

We now consider the application of the new collocation scheme in spatial discretization of time-dependent nonlinear problems. To avoid solving nonlinear systems, a semi-implicit time-marching scheme is preferred in practice. This usually leads to solving $\mathcal{L}_\lambda[u] = f$, where λ involves the time-stepping size. Thanks to (2.22), we can explicitly invert the operator by using the new basis. Accordingly, a semi-implicit scheme appears like an "explicit" scheme.

To fix the idea, we consider the Burger's equation

$$\begin{cases} \partial_t v + v\partial_y v = \nu \partial_y^2 v, & (y,t) \in (a,b) \times (0,T], \\ v(a,t) = v_-(t), \quad v(b,t) = v_+(t), & t \in [0,T], \\ v(y,0) = v_0(y), & y \in [a,b], \end{cases} \tag{5.14}$$

for constant $\nu > 0$, and given data $v_\pm, v_0$. We transform the interval of interest to the reference interval $(-1, 1)$ via

$$y = a + \frac{x+1}{2}(b-a), \quad u(x,t) = v(y,t), \quad u_0(x) = v_0(y), \quad \mu = \frac{4\nu}{(b-a)^2}, \tag{5.15}$$

and convert (5.14) into

$$\begin{cases} \partial_t u + \frac{2}{b-a} u \partial_x u = \mu \partial_x^2 u, & (x,t) \in (-1,1) \times (0,T], \\ u(-1,t) = v_-(t), \ u(1,t) = v_+(t), & t \in [0,T], \\ u(x,0) = u_0(x), & x \in [-1,1]. \end{cases} \tag{5.16}$$

Let τ be the time-stepping size and let $t_n = n\tau$ for $n = 0, 1, 2, \ldots$. We employ the Crank–Nicolson leap-frog scheme with spatial discretization by the new basis $\{Q_j\}$ (with $\lambda = 1/\sqrt{\tau\mu}$) for (5.16), that is, to find the approximation of $u(x, t_n)$ as

$$u_N^n(x) = v_-(t_n)Q_0(x) + \sum_{j=1}^{N-1} v_j^n Q_j(x) + v_+(t_n)Q_N(x) \in \mathcal{Q}_N, \quad n \geq 0,$$

such that

$$\frac{u_N^{n+1}(x_i) - u_N^{n-1}(x_i)}{2\tau} + \frac{2}{b-a} u_N^n(x_i)\partial_x u_N^n(x_i) = \mu\, \partial_x^2 \Big(\frac{u_N^{n+1} + u_N^{n-1}}{2}\Big)(x_i), \quad (5.17)$$

for $1 \leq i \leq N-1$ and $n \geq 1$. It is seen that u_N^n meets the boundary conditions exactly. By Theorems 2.3 and (5.17), we obtain the following "explicit" marching scheme:

$$\boldsymbol{v}^{n+1} = -\big(2\lambda^2 \mathbf{Q}_{\text{in}} + \boldsymbol{I}_{N-1}\big)\boldsymbol{v}^{n-1} + \boldsymbol{f}^n, \quad n \geq 1, \quad (5.18)$$

where $\boldsymbol{v}^n = (v_1^n, v_2^n, \ldots, v_{N-1}^n)^t$ and $\boldsymbol{f}^n = (f_1^n, f_2^n, \ldots, f_{N-1}^n)^t$ with

$$f_i^n = \frac{4}{\mu(b-a)}\Big(v_-(t_n)q_{i0} + \sum_{j=1}^{N-1} q_{ij} v_j^n + v_+(t_n)q_{iN}\Big)$$
$$\times\Big(v_-(t_n)q_{i0}^{(1)} + \sum_{j=1}^{N-1} q_{ij}^{(1)} v_j^n + v_+(t_n)q_{iN}^{(1)}\Big).$$

Remark 5.3 We can use a suitable second-order one-step scheme to generate $\boldsymbol{v}^1$ from $\boldsymbol{v}^0$. One finds from (2.38) that the spectral radius

$$\sigma\big(2\lambda^2 \mathbf{Q}_{\text{in}} + \boldsymbol{I}_{N-1}\big) \approx \Big|\frac{4\lambda^2 - \pi^2}{4\lambda^2 + \pi^2}\Big| < 1, \quad N \gg 1. \quad (5.19)$$

Note that $\lambda^2 = (\tau\mu)^{-1}$, so the scheme is unconditionally stable. □

In the following test, we consider (5.14) with $a = -5$, $b = 5$, $T = 12$, and the exact solution:

$$v(y,t) = \varrho\Big\{1 - \tanh\Big(\frac{\varrho}{2\nu}(y - \varrho y + 3)\Big)\Big\}, \quad \nu = 0.1, \quad \varrho = 0.5.$$

We tabulate in Table 5 the CPU time, the maximum pointwise errors at final time T, and the order of convergence for the new scheme (including the overheads for pre-computing the basis functions), BCOL in [32], and LCOL schemes. Observe from Table 5 that the new

Table 5 Comparison of three collocation schemes for first example of (5.14) with $\alpha = \beta = 0$ and $N = 200$

τ	LCOL			BCOL [32]			New scheme		
	CPU (s)	Errors	Orders	CPU (s)	Errors	Orders	CPU (s)	Errors	Orders
0.01000	34.500	1.38e+00	–	30.585	1.41e+00	–	5.039	1.41e+00	–
0.00500	49.010	2.05e−01	2.75	86.321	2.40e−02	5.88	4.971	2.42e−02	5.86
0.00100	134.875	9.06e−05	4.80	475.908	3.26e−07	6.96	6.084	3.26e−07	6.97
0.00050	170.766	9.06e−05	−0.00	1205.271	8.15e−08	2.00	7.318	8.15e−08	2.00
0.00010	476.973	9.06e−05	−0.00	9064.323	3.26e−09	2.00	15.849	3.19e−09	2.01
0.00005	1121.298	9.06e−05	−0.00	33228.976	8.16e−10	2.00	49.842	7.79e−10	2.04

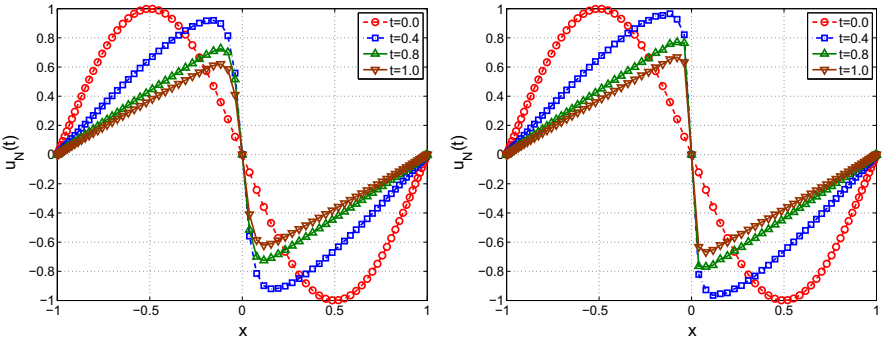

Fig. 6 Profiles of numerical solutions of (5.14) and (5.20) with $N = 80$ and $\tau = 0.01$. *Left* $\nu = 0.02$. *Right* $\nu = 0.01$

scheme is much faster for small time-stepping size or long time integration, since there is no need to solve a linear system for each time stepping.

We further test the new method by considering (5.14) with

$$v_0(x) = -\sin(\pi x), \quad v_\pm(t) = 0, \quad a = -1, \quad b = 1. \tag{5.20}$$

We plot in Fig. 6 the profiles of the numerical solutions at time $t = 0, 0.4, 0.8, 1$ for two values of $\nu = 0.02$ and $\nu = 0.01$, respectively. We clearly observe the steepening of the profiles at the center of the region, before the shock is developed.

5.3 Two-Dimensional Poisson-type Equations

We next extend the new collocation scheme to multiple dimensions. Consider the two-dimensional Poisson-type equation

$$\Delta u - 2\lambda^2 u = f, \quad (x, y) \in \Omega = (-1, 1) \times (-1, 1); \quad u|_{\partial\Omega} = 0. \tag{5.21}$$

Using the new basis $\{Q_j\}$ with parameter $\lambda > 0$, we look for the numerical solution:

$$u_N(x, y) = \sum_{m,n=1}^{N-1} \tilde{u}_{mn} Q_m(x) Q_n(y).$$

Note that by (2.8),

$$\Delta u_N - 2\lambda^2 u_N = \sum_{m,n=1}^{N-1} \tilde{u}_{mn} l_m(x) Q_n(y) + \sum_{m,n=1}^{N-1} \tilde{u}_{mn} Q_m(x) l_n(y).$$

From this property and the collocation scheme for (5.21):

$$\Delta u_N(x_i, y_j) - 2\lambda^2 u_N(x_i, y_j) = f(x_i, y_j), \quad 1 \le i, j \le N-1, \tag{5.22}$$

we obtain the linear system:

$$\boldsymbol{U}\, \boldsymbol{Q}_{\text{in}}^t + \boldsymbol{Q}_{\text{in}} \boldsymbol{U} = \boldsymbol{F}, \tag{5.23}$$

where $\boldsymbol{U} = \big(\tilde{u}_{mn}\big)_{m,n=1,\ldots,N-1}$, and $\boldsymbol{F} = \big(f(x_i, y_j)\big)_{i,j=1,\ldots,N-1}$.

Using the notion of the methods of eigen-decomposition (see, e.g., [27, Sec. 8.1.1]), we take the following steps to solve the problem:

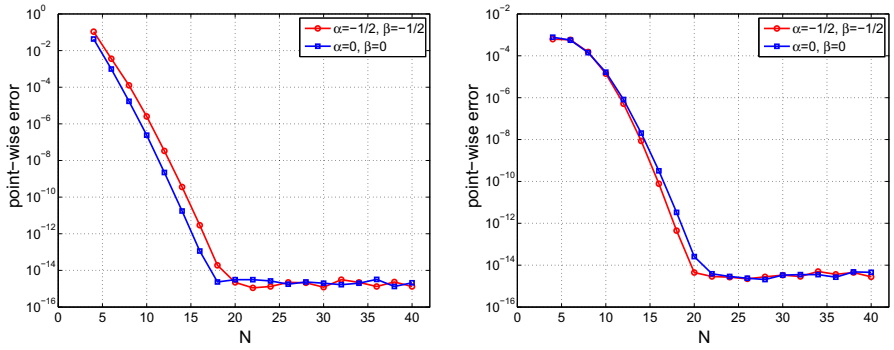

Fig. 7 Comparison of maximum pointwise errors for (5.21). *Left* $\lambda = 1$. *Right* $\lambda = 100$

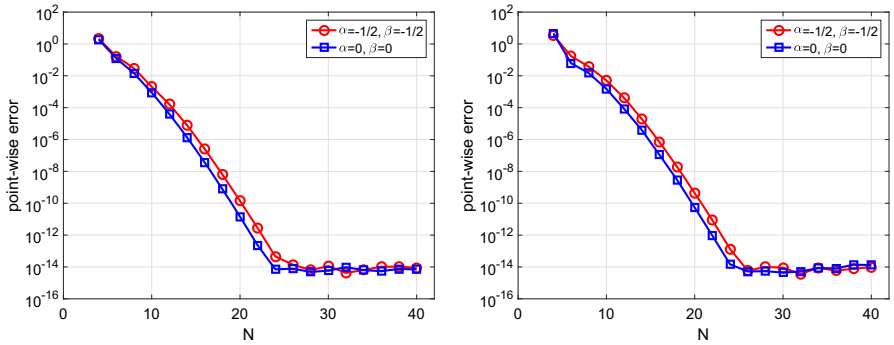

Fig. 8 Comparison of maximum pointwise errors for (5.25). *Left* $\lambda = 1$. *Right* $\lambda = 10$

1. Pre-processing: compute the eigenvalue and eigenvectors $(\boldsymbol{\Gamma}, \boldsymbol{E})$ of $\boldsymbol{Q}_{\text{in}}$;
2. Compute $\tilde{\boldsymbol{F}} = \boldsymbol{E}^{-1}\boldsymbol{F}(\boldsymbol{E}^{-1})^t$;
3. Compute $\tilde{\boldsymbol{U}}$ from following equation

$$\tilde{U}_{ij} = \frac{\tilde{F}_{ij}}{\Gamma_{ii} + \Gamma_{jj}}, \quad 1 \le i, j \le N-1; \tag{5.24}$$

4. Obtain the solution $\boldsymbol{U} = \boldsymbol{E}\tilde{\boldsymbol{U}}\boldsymbol{E}^t$. □

We test (5.21) with the exact solution $u = \sin(2\pi x)\sin(2\pi y)$. In Fig. 7, we plot the maximum pointwise errors for various N. This show that the new scheme has spectral accuracy if the solution is smooth.

Lastly, we consider the Poisson equation

$$\Delta u = f, \quad (x, y) \in \Omega = (-1, 1)^2; \quad u|_{\partial\Omega} = 0. \tag{5.25}$$

In this case, we essentially replace (5.24) by

$$\tilde{U}_{ij} = \frac{\tilde{F}_{ij}}{\Gamma_{ii} + \Gamma_{jj} + 2\lambda^2\Gamma_{ii}\Gamma_{jj}}, \quad 1 \le i, j \le N-1.$$

In Fig. 8, we depict the maximum pointwise errors for (5.21) with the same exact solution $u = \sin(2\pi x)\sin(2\pi y)$. Once again, we observe the spectral accuracy as the previous case.

6 Formulas for C_1 and C_2 in Proposition 2.1

Corresponding to the typical boundary conditions in (2.3), we have the following formulas for the constant C_1 and C_2 in Proposition 2.1:

- if $\mathscr{B}_\pm[u] = u(\pm 1) = g_\pm$, then

$$\begin{aligned} C_1 &= \frac{1}{2\sinh(2\lambda)}\big\{g_+ + \mathcal{I}_\lambda^-[f](1)\big\} - \frac{e^{-2\lambda}}{2\sinh(2\lambda)}\big\{g_- + \mathcal{I}_\lambda^+[f](-1)\big\}, \\ C_2 &= -\frac{1}{2\sinh(2\lambda)}\big\{g_+ + \mathcal{I}_\lambda^-[f](1)\big\} + \frac{e^{2\lambda}}{2\sinh(2\lambda)}\big\{g_- + \mathcal{I}_\lambda^+[f](-1)\big\}\big\}; \end{aligned} \tag{6.1}$$

- if $\mathscr{B}_\pm[u] = u'(\pm 1) = g_\pm$, then

$$\begin{aligned} C_1 &= \frac{\lambda^{-1}}{2\sinh(2\lambda)}\big\{g_+ - \lambda\mathcal{I}_\lambda^-[f](1)\big\} - \frac{\lambda^{-1}e^{-2\lambda}}{2\sinh(2\lambda)}\big\{g_- + \lambda\mathcal{I}_\lambda^+[f](-1)\big\}, \\ C_2 &= \frac{\lambda^{-1}}{2\sinh(2\lambda)}\big\{g_+ - \lambda\mathcal{I}_\lambda^-[f](1)\big\} - \frac{\lambda^{-1}e^{2\lambda}}{2\sinh(2\lambda)}\big\{g_- + \lambda\mathcal{I}_\lambda^+[f](-1)\big\}; \end{aligned} \tag{6.2}$$

- if $\mathscr{B}_-[u] = u(-1) = g_-$ and $\mathscr{B}_+[u] = u'(1) + \eta\, u(1) = g_+$, then

$$\begin{aligned} C_1 &= \frac{g_+ + (\eta-\lambda)\mathcal{I}_\lambda^-[f](1)}{2\big(\lambda\cosh(2\lambda) + \eta\sinh(2\lambda)\big)} - \frac{(\eta-\lambda)e^{-2\lambda}\big\{g_- + \mathcal{I}_\lambda^+[f](-1)\big\}}{2\big(\lambda\cosh(2\lambda) + \eta\sinh(2\lambda)\big)}, \\ C_2 &= -\frac{g_+ + (\eta-\lambda)\mathcal{I}_\lambda^-[f](1)}{2\big(\lambda\cosh(2\lambda) + \eta\sinh(2\lambda)\big)} + \frac{(\eta+\lambda)e^{2\lambda}\big\{g_- + \mathcal{I}_\lambda^+[f](-1)\big\}}{2\big(\lambda\cosh(2\lambda) + \eta\sinh(2\lambda)\big)}. \end{aligned} \tag{6.3}$$

7 Jacobi Polynomials and Jacobi–Gauss–Lobatto Quadrature

Let $P_n^{(\alpha,\beta)}(x)$ ($x \in [-1,1]$ and $\alpha, \beta > -1$) be the Jacobi polynomial of degree n, as normalized in [30]. We also refer to [30] for the following basic properties.

The Jacobi polynomials are eigenfunctions of the Sturm-Liouville equation

$$(x^2-1)\partial_x^2 P_n^{(\alpha,\beta)}(x) + \big\{\alpha - \beta + (\alpha+\beta+2)x\big\}\partial_x P_n^{(\alpha,\beta)}(x) = \lambda_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x), \tag{7.1}$$

where the corresponding eigenvalues are

$$\lambda_n^{(\alpha,\beta)} = n(n+\alpha+\beta+1). \tag{7.2}$$

The Jacobi polynomials are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$, namely,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_{n'}^{(\alpha,\beta)}(x)\omega^{(\alpha,\beta)}(x)\,\mathrm{d}x = \gamma_n^{(\alpha,\beta)}\delta_{nn'}, \tag{7.3}$$

where $\delta_{nn'}$ is the Dirac Delta symbol, and the normalization constant is given by

$$\gamma_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}. \tag{7.4}$$

We have

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x); \quad P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}. \tag{7.5}$$

Moreover, there holds the important derivative formula:

$$\partial_x P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n \geq 1. \tag{7.6}$$

We also use the following recurrent relation:

$$P_n^{(\alpha,\beta)}(x) = a_n \partial_x P_{n-1}^{(\alpha,\beta)}(x) + b_n \partial_x P_n^{(\alpha,\beta)}(x) + c_n \partial_x P_{n+1}^{(\alpha,\beta)}(x), \tag{7.7}$$

where $a_1 := a_1^{(\alpha,\beta)} = 0$, and

$$a_n := a_n^{(\alpha,\beta)} = -\frac{2(n+\alpha)(n+\beta)}{(n+\alpha+\beta)(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \quad n > 1, \tag{7.8a}$$

$$b_n := b_n^{(\alpha,\beta)} = \frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \quad n \geq 1, \tag{7.8b}$$

$$c_n := c_n^{(\alpha,\beta)} = \frac{2(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 1. \tag{7.8c}$$

The Jacobi–Gauss–Lobatto (JGL) points $\{x_j = \xi_{N,j}^{(\alpha,\beta)}\}_{j=0}^N$ (with $x_0 = -1, x_N = 1$) are zeros of $(1-x^2)\partial_x P_N^{(\alpha,\beta)}(x)$. Let $\{\omega_j = \omega_{N,j}^{(\alpha,\beta)}\}_{j=0}^N$ be the corresponding JGL quadrature weights (cf. [27, Theorem. 3.27]). Then we have

$$\int_{-1}^1 \phi(x)\psi(x)\omega^{(\alpha,\beta)}(x)\,\mathrm{d}x = \sum_{j=0}^N \phi(x_j)\psi(x_j)\omega_j, \quad \forall \phi\cdot\psi \in \mathcal{P}_{2N-1}. \tag{7.9}$$

Let $\{h_j := h_{N,j}^{(\alpha,\beta)}\}$ be the Lagrange interpolating basis polynomials associated with $\{x_j\}_{j=0}^N$, such that $h_j \in \mathcal{P}_N$ and $h_j(x_i) = \delta_{ij}$. We have the representation

$$h_j(x) = \sum_{n=0}^N \frac{\omega_j}{\tilde{\gamma}_n} P_n^{(\alpha,\beta)}(x_j) P_n^{(\alpha,\beta)}(x), \tag{7.10}$$

where

$$\tilde{\gamma}_n = \gamma_n^{(\alpha,\beta)}, \quad 0 \leq n \leq N-1; \quad \tilde{\gamma}_N = \Big(2 + \frac{\alpha+\beta+1}{N}\Big)\gamma_N^{(\alpha,\beta)}.$$

Let $\mathbb{I}_N$ be the corresponding Lagrange interpolation operator, namely, $\mathbb{I}_N : C(\bar{\Lambda}) \to \mathcal{P}_N$ such that for any $u \in C(\bar{\Lambda})$,

$$\mathbb{I}_N u(x_j) = u(x_j), \quad 0 \leq j \leq N. \tag{7.11}$$

8 Proof of Proposition 3.3

By the orthogonality (7.3) and (7.9),

$$\begin{aligned} \mu_j^n &= \frac{1}{\gamma_n^{(\alpha,\beta)}} \int_{-1}^1 l_j(x) P_n^{(\alpha,\beta)}(x)\, \omega^{(\alpha,\beta)}(x)\,\mathrm{d}x \\ &= \frac{1}{\gamma_n^{(\alpha,\beta)}} \Big\{ l_j(-1) P_n^{(\alpha,\beta)}(-1)\omega_0 + P_n^{(\alpha,\beta)}(x_j)\omega_j + l_j(1) P_n^{(\alpha,\beta)}(1)\omega_N \Big\}, \end{aligned} \tag{8.1}$$

where we used the property: $l_j(x_i) = \delta_{ij}$ for $1 \le i, j \le N-1$ (cf. (2.7)). Thus, it remains to derive the explicit formulas for $l_j(\pm 1)$. As the interior JGL points $\{x_j\}_{j=1}^{N-1}$ are zeros of $\partial_x P_N^{(\alpha,\beta)}(x)$, we have

$$l_j(\pm 1) = \frac{\partial_x P_N^{(\alpha,\beta)}(x)}{(x - x_j)\,\partial_x^2 P_N^{(\alpha,\beta)}(x_j)}\bigg|_{x=\pm 1}. \tag{8.2}$$

By (7.1),

$$\begin{aligned} &2(\beta+1)\partial_x P_N^{(\alpha,\beta)}(-1) = -\lambda_N^{(\alpha,\beta)} P_N^{(\alpha,\beta)}(-1), \quad 2(\alpha+1)\partial_x P_N^{(\alpha,\beta)}(1) = \lambda_N^{(\alpha,\beta)} P_N^{(\alpha,\beta)}(1), \\ &-(1-x_j^2)\partial_x^2 P_N^{(\alpha,\beta)}(x_j) = \lambda_N^{(\alpha,\beta)} P_N^{(\alpha,\beta)}(x_j), \quad 1 \le j \le N-1. \end{aligned} \tag{8.3}$$

A direct calculation from (8.2) and (8.3) leads to

$$l_j(-1) = -\frac{1-x_j}{2(\beta+1)} \frac{P_N^{(\alpha,\beta)}(-1)}{P_N^{(\alpha,\beta)}(x_j)}, \quad l_j(1) = -\frac{1+x_j}{2(\alpha+1)} \frac{P_N^{(\alpha,\beta)}(1)}{P_N^{(\alpha,\beta)}(x_j)}. \tag{8.4}$$

Thus, we obtain the desired formulas of $l_j(\pm 1)$ in (3.6).

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